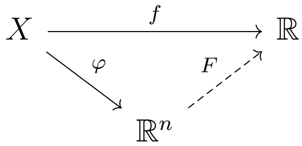
10.1. Project: The weak topology is not metrizable.

(i) With a metric, a countable neighbourhood basis can be constructed explicitly.

(ii) How does any open set in the weak topology which contains the origin look like?

(iii) Prove that if $\varphi\colon X \to \mathbb{R}^n$ and $f\colon X \to \mathbb{R}$ are linear maps such that $\ker\varphi \subset \ker f$, then there exists a linear map $F\colon \mathbb{R}^n \to \mathbb{R}$ such that $f = F \circ \varphi$. Then choose φ wisely.



(iv) If $\{A_\alpha\}_{\alpha\in\mathbb{N}}$ is a countable neighbourhood basis of $0 \in X$ in (X, τ_{w}) then each A_α contains one of the neighbourhoods of the neighbourhood basis constructed in (ii). Use this fact to construct a countable set $F \subset X^*$ such that every $f^* \in X^*$ is a linear combination of finitely many elements in F. Apply part (iii) for this last step.

(v) Recall that $(X^*, \|\cdot\|_{X^*})$ is always complete.

10.2. Non-compactness.

(i) Find a sequence of bounded functions $f_n\colon [0,1] \to \mathbb{R}$ such that for *any* pair $n, m \in \mathbb{N}$ with $n \neq m$ the difference $f_n - f_m$ is non-zero on a set of measure $\frac{1}{2}$.

(ii) The simplest sequence works.

10.3. Separability. Are subsets of separable sets separable?

How can a countable dense set in S be "scaled" to a countable dense set in X?

10.4. Quadratic functional on a reflexive space.

(i) Apply the direct method (cf. "Variationsprinzip", Satz 5.4.1).

(ii) If there exists another minimum $\bar{y}$, then $F(\frac{\bar{x}+\bar{y}}{2}) < \frac{F(\bar{x})+F(\bar{y})}{2}$. To prove that $\bar{x}$ is contained in the convex hull of $\{x_1, \ldots, x_n\}$, observe that $\frac{d}{dt}|_{t=0} F(\bar{x} + ty) = 0$ for all $y \in X$.

10.5. A class of functionals on a reflexive space. Prove that $\alpha_0 = 2$. Indeed, on one hand it is possible to apply the direct method to F_α for $0 \le \alpha < 2$. On the other hand, for $\alpha > 2$ there are easy finite dimensional examples for which $\inf_{x\in X} F_\alpha(x) = -\infty$.