11.1. Dual operators. Every statement follows directly from the property which characterises dual operators. You can apply (i) and (ii) to show (iii).

11.2. Isomorphisms and isometries. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces and $T \in L(X, Y)$. Prove the following statements.

(i) Apply Problem 11.1 (iii).

(ii) Due to part (i) it suffices to show $\|T^*y^*\|_{X^*} = \|y^*\|_{Y^*}$ for every $y^* \in Y^*$.

(iii) Combine part (i) respectively (ii) with the result of Problem 11.1 (iv).

(iv) Apply part (i) twice and reuse the result of Problem 11.1 (iv).

11.3. Operator on compact sequences.

(i) The answer is no. In particular the operator T is not bounded in c_c.

(ii) Recall that $(\ell^1)^* \cong \ell^\infty$ and $(c_0)^* \cong \ell^1$. Then prove that $A\colon (c_c, \|\cdot\|_{\ell^\infty}) \to \mathbb{R}$ given by $Ax := \sum_{n\in\mathbb{N}} y_n (Tx)_n$ is continuous if and only if $\sum_{n\in\mathbb{N}} |ny_n| < \infty$.

(iii) Show that, if $x^{(k)} \in c_c$ converges to 0 in ℓ^∞ and $Tx^{(k)}$ converges to some $y \in \ell^1$, then $y = 0$. This implies that T is closable. To find an element in $D_{\overline{T}} \setminus c_c$, consider an $x = (x_n)_{n\in\mathbb{N}} \in c_c \setminus c_0$ such that x_n goes to zero sufficiently fast as $n \to \infty$.

11.4. Compact operators.

(i) A sequence in $\overline{T(B_1(0))}$ can be approximated by a sequence in $T(B_1(0))$ which is the image of a (bounded) sequence in $B_1(0)$.

(ii) Use (i) and a diagonal sequence argument.

(iii) In finite dimensions, sets which are bounded and closed are compact.

(iv) Use (i) and continuity of T respectively S.

(v) Apply the Eberlein–Šmulian theorem.

11.5. Integral operators.

(i) Apply Hölder's inequality and Fubini's theorem.

(ii) By Fubini's theorem, $k(x, \cdot) \in L^2(\Omega)$ for almost every $x \in \Omega$. Apply the dominated convergence theorem in $L^2(\Omega)$. You may use Problem 11.4 (v) to conclude.

11.6. Operator that is (almost) injective.

(i) Assume by contradiction that there exist $x_k \in X$ with $\|x_k\|_X = 1$ and $\|Px_k\|_Y = 1/k$. Using $(*)$ and the compactness of J prove that $x_k \to x_\infty$ and find a contradiction.

(ii) Show that the unit ball in $\ker(P)$ is relatively compact, which implies that $\ker(P)$ has finite dimension. Then use Problem 7.2 to write $X = \ker(P) \oplus W$.