12.1. Uniform subconvergence ✍. Apply the Arzelà–Ascoli theorem.

12.2. Sequence with bounded Hölder norm ⧉.

(i) Apply the Arzelà–Ascoli theorem.

(ii) Use Hölder's inequality.

(iii) Use (i) to prove that T is a compact operator from X to $Y = C^0([0,1],\mathbb{R})$.

12.3. Multiplication operators on complex-valued sequences ⚙.

(i) Computing $\|Te_k\|_{\ell^2_{\mathbb{C}}}$ for $e_k = (0,\ldots,0,1,0,\ldots) \in \ell^2_{\mathbb{C}}$ yields a lower bound on $\|T\|$.

(ii) Compute the adjoint operator T^* explicitly.

(iii) To prove "$\Leftarrow$", show that the components $x_n^{(k)}$ of any bounded sequence $(x^{(k)})_{k\in\mathbb{N}}$ in $\ell^2_{\mathbb{C}}$ converge in $\mathbb{C}$ as $k \to \infty$ (up to subsequence).

12.4. A compact operator on continuous functions 🖩.

(i) Compute the integral

$$\int_a^x \frac{1}{\sqrt{x-t}}\,\mathrm{d}x.$$

(ii) Apply the Arzelà–Ascoli theorem.

(iii) Recall the definition of spectral radius and appeal to part(i).

12.5. A multiplication operator on square-integrable functions 🖩.

(i) Computing $\|T\chi\|_{L^2([a,b];\mathbb{C})}$ for characteristic functions χ supported near a respectively b yields a lower bound on $\|T\|$.

(ii) Recall that $Tf = \lambda f$ in $L^2([a,b])$ means $(Tf)(x) = \lambda f(x)$ for *almost* all $x \in [a,b]$.

(iii) Part (ii) implies that the operator $(\lambda - T)$ is injective for any $\lambda \in \mathbb{C}$. For which $\lambda \in \mathbb{C}$ is the operator $(\lambda - T)\colon L^2([a,b];\mathbb{C}) \to L^2([a,b];\mathbb{C})$ also surjective? As an example, compute f such that $(\lambda - T)f = 1 \in L^2([a,b])$.