13.1. Definitions of resolvent set. Given $\lambda \in \tilde{\rho}(A)$, prove that $(\lambda - A)\colon D_A \to X$ is surjective.

13.2. Unitary operators.

(i) Why do isometries $T \in L(H,H)$ satisfy $\langle Tx, Ty\rangle_H = \langle x, y\rangle_H$ for every $x, y \in H$?

(ii) Recall that the spectral radius of any operator T is bounded from above by $\|T\|$ and recall the statements of Satz 6.5.3.i and Satz 2.2.7.

13.3. Integral operators revisited. Recall the properties of K proven in Problem 11.5. Check that K and A are both self-adjoint operators. Conclude as in the proof of Beispiel 6.5.2.

13.4. Resolvents and spectral distance.

(i) Prove $R_\lambda^* = R_{\overline{\lambda}}$ and use that resolvents to different values commute (Satz 6.5.2).

(ii) Argue, that it suffices to show the following implication for any $\alpha \in \mathbb{C}$.

$$\inf_{\beta\in\sigma(B)} |\alpha - \beta| > \|A - B\|_{L(H,H)} \qquad \Rightarrow\ \alpha \in \rho(A).$$

Given $f_\alpha(z) = (\alpha - z)^{-1}$, the spectral mapping theorem implies $f_\alpha(\sigma(B)) = \sigma(f_\alpha(B))$. Show that normal operators R have spectral radius $r_R = \|R\|$. Apply Satz 2.2.7.

13.5. Compact operator on space decomposition.

(i) Nothing more than stating the theorem.

(ii) First note that H' and H'' are closed. Then use that the closure of the image of the unit ball of H' via A' is contained in the closure of the image of the unit ball of H via A, which is compact. The self-adjointness is easy to check.

(iii) Nothing more than the definition.

(iv) To prove that $\lambda_1 \geq \max\{\lambda_1', \lambda_1''\}$ use the Courant-Fischer characterization. For the other inequality apply the orthogonal decomposition in (ii) to an eigenvector e_1 of A relative to the first eigenvalue λ_1.

13.6. Heisenberg's uncertainty principle. Be pedantic about operator domains.

(i) Apply the Cauchy–Schwarz inequality.

(ii) To apply part (i), find symmetric operators $\tilde{A} = A - \lambda$ and $\tilde{B} = B - \mu$ satisfying

$$[A, B] = [\tilde{A}, \tilde{B}], \qquad \varsigma(A, x) = \|\tilde{A}x\|_H, \qquad \varsigma(B, x) = \|\tilde{B}x\|_H.$$

(iii) Check that $[A, B^n]$ is well-defined and prove $[A, B^n] = niB^{n-1}$ for every $n \in \mathbb{N}$.

(iv) The checklist is

$$
\begin{aligned}
\forall f \in D_P := C_0^1([0,1];\mathbb{C}): &\quad Pf \in L^2([0,1];\mathbb{C}), && (\Rightarrow P \text{ well-defined})\\
\forall f \in D_Q := L^2([0,1];\mathbb{C}): &\quad Qf \in L^2([0,1];\mathbb{C}), && (\Rightarrow Q \text{ well-defined})\\
\forall f \in D_{[P,Q]} := D_P \cap D_Q: &\quad Qf \in D_P, && (\Rightarrow [P,Q] \text{ well-defined})\\
\forall f,g \in D_P: &\quad \langle Pf, g\rangle_{L^2} = \langle f, Pg\rangle_{L^2}, && (\Rightarrow P \text{ symmetric})\\
\forall f,g \in D_Q: &\quad \langle Qf, g\rangle_{L^2} = \langle f, Qg\rangle_{L^2}, && (\Rightarrow Q \text{ symmetric})\\
\forall f \in D_{[P,Q]}: &\quad ([P,Q]f)(s) = if(s). && (\text{Heisenberg pair})
\end{aligned}
$$