1.1. Equivalent Norms ⚙.

Definition. Let X be a set. A *metric* on X is a non-negative function $d\colon X \times X \to \mathbb{R}$ that satisfies for all $x, y, z \in X$

$$d(x,y) = 0 \Leftrightarrow x = y, \qquad d(x,y) = d(y,x), \qquad d(x,z) \le d(x,y) + d(y,z).$$

We say that two metrics d and d' on X are *equivalent* if

$$\exists C > 0 \quad \forall x_1, x_2 \in X: \quad C^{-1} d'(x_1,x_2) \le d(x_1,x_2) \le C d'(x_1,x_2).$$

Let X be a vector space over $\mathbb{R}$. A *norm* on X is a non-negative function $\|\cdot\|\colon X \to \mathbb{R}$ which satisfies for all $x, y \in X$ and $\lambda \in \mathbb{R}$

$$\|x\| = 0 \Leftrightarrow x = 0, \qquad \|\lambda x\| = |\lambda|\|x\|, \qquad \|x + y\| \le \|x\| + \|y\|.$$

We say that two norms $\|\cdot\|$ and $\|\cdot\|'$ on X are *equivalent* if

$$\exists C > 0 \quad \forall x \in X: \quad C^{-1}\|x\|' \le \|x\| \le C\|x\|'.$$

Recall that a norm $\|\cdot\|$ induces a metric $d_{\|\cdot\|}$ by the formula $d_{\|\cdot\|}(x_1,x_2) = \|x_1 - x_2\|$.

(i) Let X be a finite-dimensional vector space over $\mathbb{R}$. Show that all norms on X are equivalent.

(ii) Construct two metrics on $\mathbb{R}^2$ that are *not* equivalent.

(iii) Construct a vector space X with two norms $\|\cdot\|$ and $\|\cdot\|'$ that are *not* equivalent.

Hint. Prove that $\|\cdot\|$ and $\|\cdot\|'$ are not equivalent by exhibiting a sequence $(x_n) \subset X$ that converges for $\|\cdot\|$ but not for $\|\cdot\|'$.

1.2. Intrinsic Characterisations ⚙.

Let V be a vector space over $\mathbb{R}$. Prove the following equivalences.

(i) The norm $\|\cdot\|$ is induced by a scalar product $\langle\cdot,\cdot\rangle$ (in the sense that there exists a scalar product $\langle\cdot,\cdot\rangle$ such that $\forall x \in V: \ \|x\|^2 = \langle x, x\rangle$)

$\Leftrightarrow$ the norm satisfies the *parallelogram identity*, i.e., $\forall x, y \in V:$

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Hint. If $\|\cdot\|$ satisfies the parallelogram identity, consider $\langle x, y\rangle := \frac{1}{4}\|x+y\|^2 - \frac{1}{4}\|x - y\|^2$. Prove $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$ first for $\lambda \in \mathbb{N}$, then for $\lambda \in \mathbb{Q}$ and finally for $\lambda \in \mathbb{R}$.

(ii) The metric $d(\cdot,\cdot)$ is induced by a norm $\|\cdot\|$ (in the sense that there exists a norm $\|\cdot\|$ such that $\forall x, y \in V: \ d(x,y) = \|x - y\|$)

$\Leftrightarrow$ the metric is *translation invariant* and *homogeneous*, i.e., $\forall v, x, y \in V \ \forall \lambda \in \mathbb{R}$:

$$d(x + v, y + v) = d(x,y),$$
$$d(\lambda x, \lambda y) = |\lambda| d(x,y).$$

1.3. When $L^p(\mathbb{R})$ and $\ell^p(\mathbb{N})$ are Hilbert spaces.

(i) Determine all values of $p \in [1,\infty]$ such that the Banach space $L^p(\mathbb{R})$ is actually Hilbertean (meaning that the norm $\|\cdot\|_{L^p}$ is induced by a scalar product).

(ii) Determine all values of $p \in [1,\infty]$ such that the Banach space $\ell^p(\mathbb{N})$ is actually Hilbertean (meaning that the norm $\|\cdot\|_{\ell^p}$ is induced by a scalar product).

It is advised not to forget the case $p=\infty$ in your discussion.

1.4. When a distance is induced by a norm.

(i) Let V be a vector space over $\mathbb{R}$ and let $d\colon V\times V\to\mathbb{R}$ be a distance. State necessary and sufficient conditions for $d(\cdot,\cdot)$ to be induced by a norm $\|\cdot\|$, in the sense that

$$d(v_1,v_2)=\|v_1-v_2\| \quad \forall\, v_1,v_2\in V.$$

(Note that only a statement is requested, no proof.)

(ii) Consider the vector space $C([0,\infty);\mathbb{R})$ consisting of continuous functions defined on $[0,\infty)\subset\mathbb{R}$ and attaining real values, and the distance

$$d(f_1,f_2)=\sum_{n=1}^{\infty}2^{-n}\frac{\|f_1-f_2\|_{C^0([0,n])}}{1+\|f_1-f_2\|_{C^0([0,n])}}$$

where $\|f\|_{C^0([0,n])}=\sup_{x\in[0,n]}|f(x)|$. Is d induced by a norm?

1.5. Infinite-dimensional vector spaces and separability.

(i) Let $\emptyset\neq\Omega\subset\mathbb{R}^n$ be an open set. Show that $L^p(\Omega)$ is an infinite-dimensional vector space for all $1\le p\le\infty$.

(ii) Let $(X,\mathcal{A},\mu)$ be a measure space. Recall that if X is separable and the measure μ is finite (or, more generally, σ-finite) and if $1\le p<\infty$, then the space $L^p(X,\mathcal{A},\mu)$ is separable. Roughly speaking, in the simple case when $X=(0,1)$, $\mathcal{A}=$ Borel-σ-algebra and $\mu=\mathscr{L}^1$, this relies on the fact that any element in those spaces can be arbitrarily well approximated by a function of the form

$$f=\sum_{i=1}^{k}q_i\,\chi_{B_i}\qquad\text{for } k\in\mathbb{N},\ B_i:=B_{r_i}(x_i),\ q_i\in\mathbb{Q},\ x_i\in\mathbb{Q}\cap(0,1),\ 0<r_i\in\mathbb{Q}.$$

Show that instead $\big(L^\infty((0,1)),\|\cdot\|_{L^\infty((0,1))}\big)$ is *not* separable, i.e., it does not contain a countable dense subset.

(Recall that $\|u\|_{L^\infty((0,1))}:=\inf\{K>0 \mid |u(x)|\le K \text{ for almost every } x\in(0,1)\}$.)