2.1. Statements of Baire.

Definition. Let (M,d) be a metric space and consider a subset $A \subset M$. Then, $\overline{A}$ denotes the closure, A° the interior and $A^\complement = M \setminus A$ the complement of A. We say that A is

- *dense*, if $\overline{A} = X$;
- *nowhere dense*, if $(\overline{A})^\circ = \emptyset$;
- *meagre*, if $A = \bigcup_{n\in\mathbb{N}} A_n$ is a countable union of nowhere dense sets A_n;
- *residual*, if $A^\complement$ is meagre.

Show that the following statements are equivalent.

(i) Every residual set $\Omega \subset M$ is dense in M.

(ii) The interior of every meagre set $A \subset M$ is empty.

(iii) The empty set is the only subset of M that is open and meagre.

(iv) Countable intersections of dense open sets are dense.

Hint. Show (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i). Use that subsets of meagre sets are meagre and recall that $A \subset M$ is dense $\Leftrightarrow$ $\overline{A} = M$ $\Leftrightarrow$ $(M \setminus A)^\circ = \emptyset$.

Remark. Baire's theorem states that (i), (ii), (iii), (iv) are true if (M,d) is complete.

2.2. Quick warm-up: true or false? Decide whether the following statements are true or false. If true, think of a quick proof. If false, find a simple counterexample. *(self-check: not to be handed in.)*

(i) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions $f_n \in C^0([0,1])$. If there exists $f\colon [0,1] \to \mathbb{R}$ such that $\forall x \in [0,1] : \lim_{n\to\infty} f_n(x) = f(x)$ then $f \in C^0([0,1])$.

(ii) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions $f_n \in C^0([0,1])$. If $\forall x \in [0,1]$ $\exists\, C(x) : \sup_{n\in\mathbb{N}} |f_n(x)| \leq C(x)$ then $\sup_{n\in\mathbb{N}} \sup_{x\in[0,1]} |f_n(x)| < \infty$.

(iii) The function $d\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ given by $d(x,y) = \min\{|x_1 - x_2|, |y_1 - y_2|\}$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$, is a distance.

(iv) There exists $A \subset \mathbb{R}$ such that both A and its complement $A^\complement$ are dense in $\mathbb{R}$.

(v) $\left(C^1([-1,1]), \|\cdot\|_{C^0}\right)$ is a Banach space, i.e., a complete normed space.

(vi) The complement of a 2^{nd} category set is a 1^{st} category set.

(vii) A nowhere dense set is meagre.

(viii) A meagre set is nowhere dense.

(ix) Let U be the set of fattened rationals in $\mathbb{R}$, namely

$$U := \bigcap_{j=1}^{\infty} U_j, \qquad U_j := \bigcup_{k=1}^{\infty} \Big(q_k - 2^{-(j+k+1)}, q_k + 2^{-(j+k+1)}\Big),$$

where $(q_n)_{n\in\mathbb{N}}$ is a counting of $\mathbb{Q}$. Then $U = \mathbb{Q}$.

2.3. An application of Baire ⚙. Let $f \in C^0([0,\infty))$ be a continuous function satisfying

$$\forall t \in [0,\infty): \lim_{n\to\infty} f(nt) = 0.$$

Prove that $\lim_{t\to\infty} f(t) = 0$.

Hint. Apply the Baire Lemma as in the proof of the uniform boundedness principle.

2.4. Compactly supported sequences and their ℓ^∞-completion ⚙.

Definition. We denote the space of compactly supported sequences by

$$c_c := \{(x_n)_{n\in\mathbb{N}} \in \ell^\infty \mid \exists N \in \mathbb{N}\ \forall n \geq N: \ x_n = 0\}$$

and the space of sequences converging to zero by

$$c_0 := \{(x_n)_{n\in\mathbb{N}} \in \ell^\infty \mid \lim_{n\to\infty} x_n = 0\}.$$

(i) Show that $(c_c, \|\cdot\|_{\ell^\infty})$ is *not* complete. What is the completion of this space?

(ii) Prove the strict inclusion

$$\bigcup_{p\geq 1} \ell^p \subsetneq c_0.$$

2.5. (Dis)-continuity of functions arising as pointwise limits ⚙ 💎.

Let (X,d) be a metric space, and let (f_n) be a sequence of continuous, real-valued functions $f_n\colon X \to \mathbb{R}$ assumed to be pointwise converging to a limit function f, i.e., we set

$$f(x) = \lim_{n\to\infty} f_n(x).$$

In this problem we wish to study the set of continuity points of the function f, namely the structure of the set $C := \{x \in X \mid f \text{ is continuous at } x\}$.

(i) Give an example of a space X and a sequence of continuous functions whose pointwise limit (although well-defined) is *not* continuous.

(ii) Assuming that (X,d) is complete, prove that C is residual and dense.

Hint. For every $\varepsilon > 0$, define $D_\varepsilon := \{x \in X \mid \operatorname{osc}_x(f) \geq \varepsilon\}$, where $\operatorname{osc}_x(f) := \lim_{r\to 0}\{\sup_{y\in B_r(x)} f(y) - \inf_{y\in B_r(x)} f(y)\}$ is the oscillation of f at x, and set $D := \bigcup_{j\geq 1} D_{1/j}$. Show that: (a) D is the set of discontinuity points of f, i.e. $D^\complement = C$, and (b) $D_\varepsilon^\complement$ is open and dense for all $\varepsilon > 0$. Hence conclude by applying Baire's Lemma.

(iii) Show that the Dirichlet function $f = \chi_{\mathbb{Q}}$ is not the pointwise limit of any sequence of continuous functions on the real line.