3.1. Bounded maps to Banach space. Let M be a set and let $(X, \|\cdot\|_X)$ be a Banach space. Then show that the set of bounded maps

$$B(M, X) := \left\{ f \colon M \to X \mid \sup_{m \in M} \|f(m)\|_X < \infty \right\}$$

endowed with the norm

$$|f| = \sup_{m \in M} \|f(m)\|_X$$

(as defined in class) is itself a Banach space.

3.2. Normal convergence. Let $(X, \|\cdot\|)$ be a normed vector space. Prove that the following statements are equivalent.

(i) $(X, \|\cdot\|)$ is a Banach space.

(ii) For every sequence $(x_n)_{n\in\mathbb{N}}$ in X with $\sum_{k=1}^{\infty} \|x_n\| < \infty$, the limit $\lim_{N\to\infty} \sum_{n=1}^{N} x_n$ exists.

Hint. A Cauchy sequence converges if and only if it has a convergent subsequence.

3.3. Subsets with compact boundary. Let $(X, \|\cdot\|)$ be an infinite-dimensional normed vector space and let $Z \subset X$ be a bounded subset with compact boundary. Prove that Z has empty interior, i.e., $Z^\circ = \emptyset$.

Hint. Assume that $Z^\circ \neq \emptyset$. Find a continuous map that projects the boundary ∂Z to the boundary of a ball inside Z. This will contradict the fact that the unit sphere in an infinite-dimensional normed space is non-compact.

3.4. Topological complement.

Definition. Let $(X, \|\cdot\|_X)$ be a Banach space. A subspace $U \subset X$ is called *topologically complemented* if there is a subspace $V \subset X$ such that the linear map I given by

$$I \colon (U \times V, \|\cdot\|_{U\times V}) \to (X, \|\cdot\|_X), \qquad \|(u, v)\|_{U\times V} := \|u\|_X + \|v\|_X,$$
$$(u, v) \mapsto u + v$$

is a continuous isomorphism of normed spaces with continuous inverse. In this case V is said to be a *topological complement* of U.

(i) Prove that $U \subset X$ is topologically complemented if and only if there exists a continuous linear map $P \colon X \to X$ with $P \circ P = P$ and image $P(X) = U$.

(ii) Show that a topologically complemented subspace must be closed.

3.5. Closed subspaces. Show that the subspaces

$$U = \{(x_n)_{n\in\mathbb{N}} \in \ell^1 \mid \forall n \in \mathbb{N}:\ x_{2n} = 0\},$$
$$V = \{(x_n)_{n\in\mathbb{N}} \in \ell^1 \mid \forall n \in \mathbb{N}:\ x_{2n-1} = n x_{2n}\}$$

are both closed in $(\ell^1, \|\cdot\|_{\ell^1})$ while the subspace $U \oplus V$ is not closed in $(\ell^1, \|\cdot\|_{\ell^1})$.

Hint. Prove that if any sequence $(x^{(k)})_{k\in\mathbb{N}}$ of elements $x^{(k)} = (x_n^{(k)})_{n\in\mathbb{N}} \in \ell^1$ converges to $(x_n)_{n\in\mathbb{N}}$ in ℓ^1 for $k \to \infty$, then each entry $x_n^{(k)}$ converges in $\mathbb{R}$ to x_n for $k \to \infty$. For the second claim, show $c_c \subset U \oplus V$. (Recall c_c from Problem 2.4.)