4.1. Operator norm ✍.

(i) Let $A \in L(\mathbb{R}^n, \mathbb{R}^n)$ be a linear map from $\mathbb{R}^n$ to itself. Show that the squared operator norm $\|A\|^2$ equals the largest eigenvalue of $A^\intercal A$.

(ii) Let $A \in L(\mathbb{R}^{2020}, \mathbb{R}^{2020})$ be symmetric such that there exists a basis $\mathcal{B}$ of $\mathbb{R}^{2020}$ diagonalising A with eigenvalues $\{\lambda_1, \lambda_2, \dots, \lambda_{2020}\} = \{1, 2, \dots, 2020\}$ each with multiplicity one.

Let $B \in L(\mathbb{R}^{2020}, \mathbb{R}^{2020})$ be symmetric such that there exists a basis $\mathcal{B}'$ not necessarily equal to $\mathcal{B}$ of $\mathbb{R}^{2020}$ diagonalising B with eigenvalues $\{\lambda_1, \lambda_2, \dots, \lambda_{2020}\} = \{1, 2, \dots, 2020\}$ each with multiplicity one.

Prove that the operator norm of the composition BA can be estimated by

$$\|BA\| < 4\,410\,000.$$

4.2. Right shift operator ✍. The right shift map on the space ℓ^2 is given by

$$\begin{aligned} S\colon \ell^2 &\to \ell^2 \\ (x_1, x_2, \dots) &\mapsto (0, x_1, x_2, \dots). \end{aligned}$$

(i) Show that that the map S is a continuous linear operator with norm $\|S\| = 1$.

(ii) Compute the eigenvalues and the spectral radius of S.

(iii) Show that S has a left inverse in the sense that there exists an operator $T\colon \ell^2 \to \ell^2$ with $T \circ S = \mathrm{id}\colon \ell^2 \to \ell^2$. Check that $S \circ T \neq \mathrm{id}$.

4.3. Volterra equation ✍. Let $k\colon [0,1] \times [0,1] \to \mathbb{R}$ be continuous. Show that for every $g \in C^0([0,1])$ there exists a unique $f \in C^0([0,1])$ satisfying

$$\forall t \in [0,1]: \quad f(t) + \int_0^t k(t,s) f(s)\,\mathrm{d}s = g(t).$$

Hint. Choose a space $(X, \|\cdot\|_X)$ and show that the operator $T\colon X \to X$ given by

$$(Tf)(t) = \int_0^t k(t,s) f(s)\,\mathrm{d}s$$

has spectral radius $r_T = 0$. Then apply Satz 2.2.7 (Struwe's notes).

4.4. Unbounded map and approximations ✍. As in Problem 2.4, we denote the space of compactly supported sequences by

$$c_c := \{(x_n)_{n\in\mathbb{N}} \in \ell^\infty \mid \exists N \in \mathbb{N}\ \forall n \geq N:\ x_n = 0\}$$

endowed with the norm $\|\cdot\|_{\ell^\infty}$. Consider the map

$$\begin{aligned} T\colon c_c &\to c_c \\ (x_n)_{n\in\mathbb{N}} &\mapsto (n x_n)_{n\in\mathbb{N}} \end{aligned}$$

(i) Show that T is not continuous.

(ii) Construct continuous linear maps $T_m\colon c_c \to c_c$ such that

$$\forall x \in c_c: \quad T_m x \xrightarrow{m\to\infty} Tx.$$

4.5. Continuity of bilinear maps ⚙. Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be normed spaces. We consider the space $(X \times Y, \|\cdot\|_{X\times Y})$, where $\|(x,y)\|_{X\times Y} = \|x\|_X + \|y\|_Y$ and a bilinear map $B\colon X \times Y \to Z$.

(i) Show that B is continuous if

$$\exists C > 0 \quad \forall (x,y) \in X \times Y: \quad \|B(x,y)\|_Z \le C\|x\|_X\|y\|_Y. \tag{†}$$

(ii) Assume that $(X, \|\cdot\|_X)$ is complete. Assume further that the maps

$$\begin{aligned} X &\to Z \\ x &\mapsto B(x,y') \end{aligned} \qquad\qquad \begin{aligned} Y &\to Z \\ y &\mapsto B(x',y) \end{aligned}$$

are continuous for every $x' \in X$ and $y' \in Y$. Prove that then, (†) holds.

Hint. Apply the Theorem of Banach-Steinhaus to a suitable map but recall that it requires completeness of the domain.