5.1. Quotient of a Hilbert space ✍. Let $(X, \langle\cdot,\cdot\rangle)$ be a Hilbert space over $\mathbb{C}$ and let $Y \subset X$ be a closed subspace. Prove that the quotient X/Y is isometric to the orthogonal $Y^{\perp}$ of Y.

5.2. Isomorphic proper subspaces ✍. Let $(X, \langle\cdot,\cdot\rangle)$ be a Hilbert space over $\mathbb{C}$ endowed with a countable Hilbertian basis $\{e_i\}_{i\geq 1}$.

(i) For all $k \geq 1$, consider the subspace $E_k \subset X$ generated by $e_1, e_3, \ldots, e_{2k-1}$, namely $E_k := \langle e_1, e_3, \ldots, e_{2k-1}\rangle_{\mathbb{C}}$ and define $Y := \cup_{k\geq 1} E_k$. Is Y closed?

(ii) Construct a proper subspace $Z \subsetneq X$ such that there exists an isomorphism $T\colon Z \to X$ of Banach spaces.

5.3. Odd and even functions ⚙. Consider the Hilbert space $H = L^2((-1,1);\mathbb{R})$ and define the subset of odd functions $D := \{f \in H \mid f(-x) = -f(x) \text{ for a.e. } x \in (-1,1)\}$ and the subset of even functions $P := \{f \in H \mid f(-x) = f(x) \text{ for a.e. } x \in (-1,1)\}$.

(i) Prove that D and P are closed subspaces of H.

(ii) Prove that $H = D \oplus P$ and $D \perp P$. Hence deduce that $D^{\perp} = P$ and $P^{\perp} = D$.

(iii) Compute the orthogonal projections $\pi_D\colon H \to D$ and $\pi_P\colon H \to P$.

(iv) Find a Hilbertian basis for both D and P.

5.4. Notable series ▦. Specifying Parseval's identity to $f(x) = x$ (seen as an element of $L^2((-\pi,\pi);\mathbb{R})$) show that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

5.5. Closed sum of subspaces ✍. Let $(X, \|\cdot\|_X)$ be a normed space an let $U, V \subset X$ be subspaces. Prove the following.

(i) If U is finite dimensional and V closed, then $U + V$ is a closed subspace of X.

(ii) If V is closed with finite codimension, i.e., $\dim(X/V) < \infty$, then $U + V$ is closed.

Hint. Is the canonical quotient map $\pi\colon X \to X/V$ continuous? What is $\pi^{-1}(\pi(U))$?

5.6. Vanishing boundary values ✍. Let $X = C^0([0,1])$ and $U = C^0_0([0,1]) := \{f \in C^0([0,1]) \mid f(0) = 0 = f(1)\}$.

(i) Show that U is a closed subspace of X endowed with the norm $\|\cdot\|_X = \|\cdot\|_{C^0([0,1])}$.

(ii) Compute the dimension of the quotient space X/U and find a basis for X/U.