6.1. 1D "closed graph theorem" for continuous functions ✍. Let $f\colon \mathbb{R}\to\mathbb{R}$ be bounded and let $\Gamma=\{(x,y)\in\mathbb{R}^2 \mid y=f(x)\}\subset\mathbb{R}^2$ be its graph. Show that f is continuous if and only if Γ is closed. (Achtung: the point here is that we are not restricting to linear maps.)

Is the same statement true if f is *not* assumed to be bounded?

6.2. Closed range ✍. Let $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ be Banach spaces. Let $A\colon D_A\subset X\to Y$ be a linear operator with closed graph. Show that the following statements are equivalent.

(i) A is injective and its range $W_A:=A(D_A)$ is closed in $(Y,\|\cdot\|_Y)$.

(ii) $\exists C>0\quad \forall x\in D_A:\quad \|x\|_X\le C\|Ax\|_Y$.

Hint. One implication follows from the Inverse Mapping Theorem.

6.3. An implication of Hellinger–Töplitz (coercive operators) ✍. Let $(H,(\cdot,\cdot))$ be a Hilbert space and let $A\colon H\to H$ be a symmetric linear operator such that

$$\exists\lambda>0\quad\forall x\in H:\quad (Ax,x)\ge\lambda\|x\|^2.$$

(Any linear operator satisfying such an inequality is called *coercive*.) Show that A is an isomorphism of normed spaces and $\|A^{-1}\|\le\lambda^{-1}$.

6.4. Graph norm ⚙.

(i) Let $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ be Banach spaces and let $A\colon D_A\subset X\to Y$ be a linear operator with graph $\Gamma_A\subset X\times Y$. Then, $\|x\|_{\Gamma_A}:=\|x\|_X+\|Ax\|_Y$ defined on D_A is called the *graph norm*.

Show that if A has closed graph, then $(D_A,\|\cdot\|_{\Gamma_A})$ is a Banach space.

(ii) Let $(X_0,\|\cdot\|_{X_0})$, $(X_1,\|\cdot\|_{X_1})$ and $(X_2,\|\cdot\|_{X_2})$ be Banach spaces and let

$$T_1\colon D_1\subset X_0\to X_1,$$
$$T_2\colon D_2\subset X_0\to X_2$$

be linear operators with closed graphs such that $D_1\subset D_2$. Prove that

$$\exists C>0\quad\forall x\in D_1:\quad \|T_2x\|_{X_2}\le C\Big(\|T_1x\|_{X_1}+\|x\|_{X_0}\Big).$$

6.5. Closed sum ⚙. Let $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ be Banach spaces and let

$$A\colon D_A\subset X\to Y,$$
$$B\colon D_B\subset X\to Y$$

be linear operators with $D_A \subset D_B$. Under the assumption that there exist constants $0 \leq a < 1$ and $b \geq 0$ such that

$$\forall x \in D_A\colon \quad \|Bx\|_Y \leq a\|Ax\|_Y + b\|x\|_X,$$

show that if A has closed graph, then $(A+B)\colon D_A \to Y$ has closed graph.

Hint. Given a sequence $(x_n)_{n\in\mathbb{N}}$ in D_A, prove the estimate

$$(1-a)\|A(x_n - x_m)\| \leq \|(A+B)(x_n - x_m)\| + b\|x_n - x_m\|.$$

6.6. Derivative operator ⚙. Let $X = L^2([0,1])$. On $D_A := C_c^\infty((0,1)) \subset X$ we define the derivative operator

$$\begin{aligned} A\colon D_A &\to X \\ f &\mapsto f'. \end{aligned}$$

Recall that A is closable. Show that the domain $D_{\overline{A}}$ of its closure is contained in

$$\{f \in C^0([0,1]) \mid f(0) = 0 = f(1)\}.$$

Hint. Given $f \in D_{\overline{A}}$ consider a sequence $(f_n)_{n\in\mathbb{N}}$ in D_A which converges to f in X. *Achtung*, L^2-convergence does *not* imply pointwise convergence: You cannot evaluate f at points. Instead, compare $f_n(t)$ to $g(t) := \int_0^t \overline{A}f \,\mathrm{d}x$.

6.7. Closable inverse ✎. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces. Let $A\colon D_A \subset X \to Y$ be a closable linear operator. Assume that its closure $\overline{A}$ is injective. Show that the inverse operator A^{-1} is closable and $\overline{A^{-1}} = (\overline{A})^{-1}$.

Hint. Consider the image of the graph of A under the map

$$\begin{aligned} \chi\colon X \times Y &\to Y \times X \\ (x,y) &\mapsto (y,x). \end{aligned}$$

6.8. Derivative operator on different spaces ✎. For the Banach spaces

(i) $X = Y = C^0([0,1])$ with norm $\|\cdot\|_{C^0([0,1])}$

(ii) $X = Y = L^2([0,1])$ with norm $\|\cdot\|_{L^2([0,1])}$

of functions $f\colon [0,1] \to \mathbb{R}$, $t \mapsto f(t)$, we consider the linear operator

$$\frac{\mathrm{d}}{\mathrm{d}t}\colon C^1([0,1]) \subset X \to Y.$$

In both cases, discuss whether this operator is bounded and whether it is closable.

Remark. Given normed spaces X and Y, a linear map $A\colon X \to Y$ is continuous or, equivalently, bounded if it satisfies one (hence all) of the conditions given in Satz 2.2.1.