D-MATH — Prof. A. Carlotto — ETH Zürich

Functional Analysis I
Problem set 7

7.1. Dense kernel. Let $(X, \|\cdot\|_X)$ be a normed space and let $f\colon X \to \mathbb{R}$ be linear and not identically zero. Show that f is *not* continuous if and only if $\ker(f)$ is dense in X.

7.2. Complementing subspaces of finite dimension or codimension. Let $(X, \|\cdot\|_X)$ be a Banach space and $U \subset X$ a closed subspace. Prove that:

(i) if $\dim(U) = n < \infty$, then U is topologically complemented;

(ii) if $\dim(X/U) = m < \infty$, then U is topologically complemented.

Remark. The notion of topological complement was defined in Problem 3.4. There we showed that $U \subset X$ is topologically complemented if (and only if) there exists a continuous linear map $P\colon X \to X$ with $P \circ P = P$ and image $P(X) = U$.

7.3. Attaining the distance from the kernel. Let $(X, \|\cdot\|_X)$ be a normed space and let $\varphi\colon X \to \mathbb{R}$ be a continuous linear functional. Assume $N := \ker(\varphi) \subsetneq X$ and let $x_0 \in X \setminus N$. Prove that the following statements are equivalent.

(i) There exists $y_0 \in N$ with $\|x_0 - y_0\|_X = \operatorname{dist}(x_0, N)$.

(ii) There exists $x_1 \in X$ with $\|x_1\|_X = 1$ and $\|\varphi\| = |\varphi(x_1)|$.

7.4. Not attaining the distance from the kernel. Consider the Banach space $(X, \|\cdot\|_X) = \big(C^0([-1,1]), \|\cdot\|_{C^0([-1,1])}\big)$. Consider the map $\varphi\colon X \to \mathbb{R}$ given by

$$\varphi(f) = \int_0^1 f(t)\,\mathrm{d}t - \int_{-1}^0 f(t)\,\mathrm{d}t.$$

(i) Show that $\varphi \in L(X, \mathbb{R})$ with $\|\varphi\|_{L(X,\mathbb{R})} = 2$.

(ii) Prove that there does not exist $f \in X$ with $\|f\|_X = 1$ and $|\varphi(f)| = 2$.

Now consider the kernel $N := \{f \in X \mid \varphi(f) = 0\}$ of φ and some $g_0 \in X \setminus N$.

(iii) Show that N is closed.

(iv) Show that there does not exist any $f_0 \in N$ such that $\|f_0 - g_0\|_X = \operatorname{dist}(g_0, N)$.

7.5. Unique extension of functionals on Hilbert spaces. Let $(H, (\cdot,\cdot)_H))$ be a Hilbert space. Let $Y \subset H$ be any subspace and let $f\colon Y \to \mathbb{R}$ be a continuous linear functional. The Hahn-Banach Theorem allows an extension $F\colon H \to \mathbb{R}$ with $F|_Y = f$ and $\|F\| = \|f\|$. Prove that F is unique.

7.6. Distance from convex sets in Hilbert spaces. Let $(H, (\cdot,\cdot)_H)$ be a Hilbert space and $\emptyset \neq Q \subset H$ a convex subset. Let $x \in H$ with distance $d := \operatorname{dist}(x, Q)$ from Q. Prove the following statements.

(i) Every sequence $(x_n)_{n\in\mathbb{N}}$ in Q with $\lim_{n\to\infty} \|x_n - x\|_H = d$ is a Cauchy sequence in H.

(ii) Suppose Q is closed in H. Then there exists a unique $y \in Q$ with $\|x - y\|_H = d$.