8.1. Duality of sequence spaces. Consider the spaces

$$c_0 := \Big\{(x_k)_{k\in\mathbb{N}} \in \ell^\infty \;\Big|\; \lim_{k\to\infty} x_k = 0\Big\}, \qquad c := \Big\{(x_k)_{k\in\mathbb{N}} \in \ell^\infty \;\Big|\; \lim_{k\to\infty} x_k \text{ exists}\Big\}.$$

(i) Quick warm-up: Is $(c_0, \|\cdot\|_{\ell^\infty})$ a Banach space? Is $(c, \|\cdot\|_{\ell^\infty})$ a Banach space?

(ii) Show that the dual space of $(c_0, \|\cdot\|_{\ell^\infty})$ is *isometrically* isomorphic to $(\ell^1, \|\cdot\|_{\ell^1})$.

(iii) To which space is the dual space of $(c, \|\cdot\|_{\ell^\infty})$ isomorphic?

8.2. A result by Lions-Stampacchia. Let $(H, (\cdot,\cdot)_H)$ be a Hilbert space and let $\emptyset \neq K \subset H$ be a closed, convex subset. Let $f\colon H \to \mathbb{R}$ be a continuous linear functional and let $a\colon H \times H \to \mathbb{R}$ be a bilinear map satisfying

(i) $\forall x, y \in H : \quad a(x,y) = a(y,x)$

(ii) $\exists \Lambda > 0 \quad \forall x, y \in H : \quad |a(x,y)| \leq \Lambda \|x\|_H \|y\|_H$

(iii) $\exists \lambda > 0 \quad \forall x \in H : \quad a(x,x) \geq \lambda \|x\|_H^2.$

Consider the functional $J\colon H \to \mathbb{R}$ given by $J(x) = a(x,x) - 2f(x)$ and prove that there exists a unique $y_0 \in K$ such that the two following inequalities both hold:

$$\begin{aligned} \forall y \in K : &\quad J(y_0) \leq J(y), \\ \forall y \in K : &\quad a(y_0, y - y_0) \geq f(y - y_0). \end{aligned}$$

Moreover show that y_0 is equal to Px_0, where $x_0 \in H$ is such that $f(x) = a(x_0, x)$ and $P\colon H \to K$ is the operator mapping $x \in H$ to the unique point $Px \in K$ with $\|x - Px\|_H = \operatorname{dist}(x, K)$ (see Problem 7.6 (ii)).

8.3. Projection to convex sets. Let $(H, (\cdot,\cdot)_H)$ be a Hilbert space and let $\emptyset \neq K \subset H$ be a closed, convex subset. Let $P\colon H \to K$ be the operator which maps $x \in H$ to the unique point $Px \in K$ with $\|x - Px\|_H = \operatorname{dist}(x, K)$ (see Problem 7.6 (ii)).

(i) For every $x_1, x_2 \in H$ prove the inequality

$$\|Px_1 - Px_2\|_H \leq \|x_1 - x_2\|_H.$$

Hint. Use Problem 8.2.

(ii) Prove that

$$K = \bigcap_{x\in H} \{y \in H \mid (Px - x, y - Px)_H \geq 0\}.$$

8.4. Strict convexity.

Definition. A normed space $(X, \|\cdot\|_X)$ is called *strictly convex* if $\|\lambda x + (1-\lambda)y\|_X < 1$ holds for all $0 < \lambda < 1$ and all $x, y \in X$ with $x \neq y$ and $\|x\|_X = 1 = \|y\|_X$.

Let $(X, \|\cdot\|_X)$ be a normed space. The "abundance"-Lemma (Satz 4.2.1) states that

$$\forall x \in X \quad \exists x^* \in X^* : \quad \|x^*\|_{X^*}^2 = x^*(x) = \|x\|_X^2.$$

(i) Prove that if X^* (but not necessarily X) is strictly convex, then for all $x \in X$ there exists a *unique* $x^* \in X^*$ with $\|x^*\|_{X^*}^2 = x^*(x) = \|x\|_X^2$.

(ii) Find a counterexample for uniqueness of such x^*, if X^* is not strictly convex.

8.5. Uniform convexity ✍.

Definition. Let $(X, \|\cdot\|_X)$ be a Banach space and let $S = \{x \in X \mid \|x\|_X = 1\}$ be the unit sphere in X. The space $(X, \|\cdot\|_X)$ is called *uniformly convex* if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x, y \in S : \quad \|x - y\|_X > \varepsilon \implies \left\|\frac{x+y}{2}\right\|_X < 1 - \delta.$$

Remark. Uniform convexity is not to be confused with *strict convexity* defined in Problem 8.4.

(i) Prove that Hilbert spaces are uniformly convex.

(ii) Provide an example of a Banach space which is not uniformly convex.

8.6. Functional on the span of a sequence ✍. Let $(X, \|\cdot\|_X)$ be a normed space, let $(x_k)_{k\in\mathbb{N}}$ be a sequence in X and $(\alpha_k)_{k\in\mathbb{N}}$ a sequence in $\mathbb{R}$. Prove that the following statements are equivalent.

(i) There exists $\ell \in X^*$ satisfying $\ell(x_k) = \alpha_k$ for every $k \in \mathbb{N}$.

(ii) There exists $\gamma > 0$ such that for every sequence $(\beta_k)_{k\in\mathbb{N}}$ in $\mathbb{R}$ and every $n \in \mathbb{N}$ it holds

$$\left|\sum_{k=1}^{n} \beta_k \alpha_k\right| \leq \gamma \left\|\sum_{k=1}^{n} \beta_k x_k\right\|_X.$$