9.1. Representation of a convex set. Let $(X, \|\cdot\|_X)$ be a normed space and let $\emptyset \neq Q \subset X$ be an open, convex subset containing the origin. Prove that there exists a subset $\Upsilon \subset X^*$ such that

$$Q = \bigcap_{f \in \Upsilon} \{x \in X \mid f(x) < 1\},$$

which means that Q is an intersection of open, affine half-spaces.

9.2. Extremal subsets.

Definition. Let X be a vector space and $K \subset X$ any subset. A subset $M \subset K$ is called *extremal subset* of K if

$$\forall x_1, x_0 \in K \quad \forall \lambda \in (0,1): \quad \Bigl(\lambda x_1 + (1-\lambda)x_0 \in M \;\Rightarrow\; x_1, x_0 \in M\Bigr)$$

If M consists of only one point $M = \{y\}$, we say that y is an *extremal point* of K.

Let X be vector space and let $K \subset X$ be a convex subset with more than one element.

(i) Assume $K \subset \mathbb{R}^2$ is also closed. Prove that the set E of all extremal points of K is closed.

(ii) Is the statement of (i) also true in $\mathbb{R}^3$?

(iii) Given an extremal subset $M \subset K$ of K, prove that $K \setminus M$ is convex.

(iv) Prove that $y \in K$ is an extremal point of K if and only if $K \setminus \{y\}$ is convex.

(v) If $N \subset K$ and $K \setminus N$ are both convex, does it follow that N is extremal?

9.3. Weak sequential continuity of linear operators. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces and let $T\colon X \to Y$ be a linear operator. Prove that the following statements are equivalent.

(i) T is continuous.

(ii) For every sequence $(x_n)_{n\in\mathbb{N}}$ in X, weak convergence $x_n \xrightarrow{w} x$ in X for $n \to \infty$ implies weak convergence $Tx_n \xrightarrow{w} Tx$ in Y for $n \to \infty$.

9.4. Weak convergence in finite dimensions. Let $(X, \|\cdot\|_X)$ be a normed space of *finite* dimension $\dim X = d < \infty$. Let $x \in X$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Prove that weak convergence $x_n \xrightarrow{w} x$ for $n \to \infty$ implies $\|x_n - x\|_X \to 0$ for $n \to \infty$.

9.5. Weak convergence in Hilbert spaces. Let $(H, (\cdot,\cdot)_H)$ be a real, infinite dimensional Hilbert space. Let $x \in H$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in H.

(i) Prove that weak convergence $x_n \xrightarrow{w} x$ in H and convergence of the norms $\|x_n\|_H \to \|x\|_H$ in $\mathbb{R}$ implies (strong) convergence $x_n \to x$ in H, i.e. $\|x_n - x\|_H \to 0$.

(ii) Suppose $x_n \xrightarrow{w} x$ and $\|y_n - y\|_H \to 0$, where $(y_n)_{n\in\mathbb{N}}$ is another sequence in H and $y \in H$. Prove that $(x_n, y_n)_H \to (x, y)_H$.

(iii) Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal system of $(H, (\cdot,\cdot)_H)$. Prove $e_n \xrightarrow{w} 0$ as $n \to \infty$.

(iv) Given any $x \in H$ with $\|x\|_H \leq 1$, prove that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in H satisfying $\|x_n\|_H = 1$ for all $n \in \mathbb{N}$ and $x_n \xrightarrow{w} x$ as $n \to \infty$.

(v) Let the functions $f_n\colon [0, 2\pi] \to \mathbb{R}$ be given by $f_n(t) = \sin(nt)$ for $n \in \mathbb{N}$. Prove that $f_n \xrightarrow{w} 0$ in $L^2([0, 2\pi])$ as $n \to \infty$.

9.6. Sequential closure ✍. Let X be a set and τ a topology on X. Given a subset $\Omega \subset X$, we use the notation

$$\overline{\Omega}_\tau := \bigcap_{\substack{A\supset\Omega,\\ X\setminus A\in\tau}} A$$

for the closure of Ω in the topology τ and

$$\overline{\Omega}_{\tau\text{-seq}} := \{x \in X \mid \exists (x_n)_{n\in\mathbb{N}} \text{ in } \Omega :\ x_n \xrightarrow{\tau} x \text{ as } n \to \infty\}$$

for the sequential closure of Ω induced by the topology τ, which is based on the notion of convergence in topological spaces:

$$(x_n \xrightarrow{\tau} x) \quad \Leftrightarrow \quad (\forall U \in \tau,\ x \in U \quad \exists N \in \mathbb{N} \quad \forall n \geq N: \quad x_n \in U).$$

(i) Prove that if $A \subset X$ is closed, then A is sequentially closed. Deduce the inclusion $\overline{\Omega}_{\tau\text{-seq}} \subset \overline{\Omega}_\tau$ for any subset $\Omega \subset X$.

(ii) Let $(X, \tau) = (\ell^2, \tau_{\mathrm{w}})$, where τ_{w} denotes the weak topology on ℓ^2. Find a set $\Omega \subset \ell^2$ for which the inclusion $\overline{\Omega}_{\text{w-seq}} \subset \overline{\Omega}_{\mathrm{w}}$ proven in (i) is strict. 💎

9.7. Convex hull ⚙.

Definition. Let $(X, \|\cdot\|_X)$ be a normed space. The *convex hull* of $A \subset X$ is defined as

$$\operatorname{conv}(A) := \bigcap_{\substack{B\supset A,\\ B \text{ convex}}} B$$

Recall the following representation theorem for convex hulls

$$\operatorname{conv}(A) = \Bigl\{\sum_{k=1}^{n} \lambda_k x_k \Bigm| n \in \mathbb{N},\ x_1, \ldots, x_n \in A,\ \lambda_1, \ldots, \lambda_n \geq 0,\ \sum_{k=1}^{n} \lambda_k = 1\Bigr\}.$$

(i) Using the representation of the convex hull above, prove Mazur's Lemma: If $(x_k)_{k\in\mathbb{N}}$ is a sequence in X satisfying $x_k \xrightarrow{w} x$ as $k \to \infty$, then there exists a sequence $(y_n)_{n\in\mathbb{N}}$ of convex linear combinations

$$y_n = \sum_{k=1}^{c(n)} a_{kn} x_k, \quad c(n) \in \mathbb{N}, \quad a_{kn} \geq 0 \text{ for } k = 1, \ldots, c(n), \quad \sum_{k=1}^{c(n)} a_{kn} = 1,$$

such that $\|y_n - x\|_X \to 0$ as $n \to \infty$.

(ii) Let $(X, \|\cdot\|_X)$ be a normed space and let $A, B \subset X$ be compact, convex subsets. Using the representation of the convex hull above, prove that $\operatorname{conv}(A \cup B)$ is compact.