10.1. Project: The weak topology is not metrizable.

Definition. Let (X,τ) be a topological space. Denoting the set of all neighbourhoods of a point $x \in X$ by

$$\mathcal{U}_x = \{U \subset X \mid \exists\, O \in \tau : \ x \in O \subset U\},$$

we say that $\mathcal{B}_x \subset \mathcal{U}_x$ is a *neighbourhood basis* of x in (X,τ) if $\forall\, U \in \mathcal{U}_x \ \ \exists V \in \mathcal{B}_x : \ V \subset U$.

Definition. A topological space (X,τ) is called *metrizable* if there exists a metric (namely a distance function) $d\colon X \times X \to \mathbb{R}$ on X (as defined in Problem 1.1) such that, denoting $B_\varepsilon(x) = \{y \in X \mid d(x,y) < \varepsilon\}$, there holds

$$\tau = \{O \subset X \mid \forall x \in O\ \exists\, \varepsilon > 0 : \ B_\varepsilon(x) \subset O)\}.$$

(i) Show that any metrizable topology τ satisfies the *first axiom of countability* which means that each point has a *countable* neighbourhood basis.

From now on, let us assume that $(X, \|\cdot\|_X)$ is a normed space and τ_{w} denotes the weak topology on X.

(ii) Prove that

$$\mathcal{B} := \left\{ \bigcap_{k=1}^{n} f_k^{-1}\Big((-\varepsilon,\varepsilon)\Big) \ \middle|\ n \in \mathbb{N},\ f_1,\ldots,f_n \in X^*,\ \varepsilon > 0 \right\}$$

is a neighbourhood basis of $0 \in X$ in (X,τ_{w}).

(iii) Prove the following lemma: Let $f_1,\ldots,f_n \in X^*$ and $f \in X^*$ be given. Let

$$N := \{x \in X \mid f_1(x) = \ldots = f_n(x) = 0\}.$$

Then $f(x) = 0$ for every $x \in N$ if and only if $f = \lambda_1 f_1 + \ldots + \lambda_n f_n$ for some $\lambda_1,\ldots,\lambda_n \in \mathbb{R}$.

(iv) Using (ii) and (iii), show that if (X,τ_{w}) is first countable, then $(X^*, \|\cdot\|_{X^*})$ admits a countable algebraic basis.

(v) Assume that the normed space $(X,\|\cdot\|_X)$ is infinite dimensional and conclude from (i) and (iv) (recalling also that any algebraic basis of a Banach space is either finite or uncountable) that the topological space (X,τ_{w}) is not metrizable.

10.2. Non-compactness. In each of the Banach spaces below, find a sequence which is bounded but does not have a convergent subsequence.

(i) $\Big(L^p([0,1]), \|\cdot\|_{L^p([0,1])}\Big)$ for $1 \le p \le \infty$.

(ii) $(c_0, \|\cdot\|_{\ell^\infty})$ where $c_0 \subset \ell^\infty$ is the space of sequences converging to zero.

10.3. Separability ✍**.** Let $(X, \|\cdot\|_X)$ be a normed space. Prove that the following statements are equivalent.

(i) The normed space $(X, \|\cdot\|_X)$ is separable.

(ii) $B = \{x \in X \mid \|x\|_X \leq 1\}$ is separable.

(iii) $S = \{x \in X \mid \|x\|_X = 1\}$ is separable.

10.4. Quadratic functional on a reflexive space ⧉**.** Let $(X, \|\cdot\|_X)$ be a reflexive Banach space over $\mathbb{R}$. Given a positive integer n, consider n pairwise distinct points $x_1, \ldots, x_n$ in X and the functional

$$F\colon X \to \mathbb{R}, \qquad F(x) = \sum_{i=1}^{n} \|x - x_i\|_X^2.$$

(i) Prove that the functional F has a global minimum on X, namely the value $\inf_{x \in X} F(x)$ is a real number attained by F at some $\overline{x} \in X$.

(ii) Let us now assume that $(X, \|\cdot\|_X)$ is a Hilbert space (thus $\|\cdot\|_X$ is induced by a scalar product $\langle \cdot, \cdot \rangle_X$). Prove that the minimum $\overline{x} \in X$ is unique, and that $\overline{x}$ belongs to the convex hull K of $\{x_1, \ldots, x_n\}$.

10.5. A class of functionals on a reflexive space ⧉**.** Let $(X, \|\cdot\|_X)$ be a reflexive Banach space over $\mathbb{R}$. Let $\ell \in X^*$ and for any given real number $\alpha \geq 0$ consider the functional $F_\alpha\colon X \to \mathbb{R}$ given by

$$F_\alpha(x) = \|x\|_X^2 - |\ell(x)|^\alpha.$$

Prove that there exists $\alpha_0 > 0$ (to be explicitly determined) such that:

(i) For any $0 \leq \alpha < \alpha_0$ the functional F_α has a global minimum on X, namely the value $\inf_{x \in X} F_\alpha(x)$ is a real number attained by F_α at some (not necessarily unique) $\overline{x} \in X$.

(ii) For any $\alpha > \alpha_0$ there exist examples of reflexive Banach spaces $(X, \|\cdot\|_X)$ and linear functionals $\ell \in X^*$ such that one has instead $\inf_{x \in X} F_\alpha(x) = -\infty$.