11.1. Dual operators. Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be normed spaces. Recall that if $T \in L(X, Y)$, then its dual operator T^* is in $L(Y^*, X^*)$ and it is characterised by the property

$$\forall x \in X \quad \forall y^* \in Y^* : \quad \langle T^* y^*, x\rangle_{X^* \times X} = \langle y^*, Tx\rangle_{Y^* \times Y}.$$

Prove the following facts about dual operators.

(i) $(\mathrm{Id}_X)^* = \mathrm{Id}_{X^*}$

(ii) If $T \in L(X, Y)$ and $S \in L(Y, Z)$, then $(S \circ T)^* = T^* \circ S^*$.

(iii) If $T \in L(X, Y)$ is bijective with inverse $T^{-1} \in L(Y, X)$, then $(T^*)^{-1} = (T^{-1})^*$.

(iv) Let $\mathcal{I}_X : X \hookrightarrow X^{**}$ and $\mathcal{I}_Y : Y \hookrightarrow Y^{**}$ be the canonical inclusions. Then,

$$\forall\, T \in L(X, Y) : \quad \mathcal{I}_Y \circ T = (T^*)^* \circ \mathcal{I}_X.$$

11.2. Isomorphisms and isometries. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces and $T \in L(X, Y)$. Prove the following statements.

(i) If T is an isomorphism, then T^* is an isomorphism.

(ii) If T is an isometric isomorphism, then T^* is an isometric isomorphism.

(iii) If X and Y are both reflexive, then the reverse implications of (i) and (ii) hold.

(iv) If $(X, \|\cdot\|_X)$ is a reflexive Banach space isomorphic to the normed space $(Y, \|\cdot\|_Y)$, then Y is reflexive.

11.3. Operator on compact sequences. Consider the space $(c_0, \|\cdot\|_{\ell^\infty})$, where as usual $c_0 := \{(x_n)_{n\in\mathbb{N}} \in \ell^\infty \mid \lim_{n\to\infty} x_n = 0\}$ and the subspace $c_c := \{(x_n)_{n\in\mathbb{N}} \in \ell^\infty \mid \exists N \in \mathbb{N}\ \forall n \geq N : \ x_n = 0\}$. Consider the linear operator

$$T : c_c \subset c_0 \to \ell^1, \qquad\qquad (Tx)_n = n x_{n+1}.$$

(i) Is T extendable to a bounded linear operator $T : c_0 \to \ell^1$? Justify your answer.

(ii) Compute the adjoint of T, namely determine

$$T^* : D_{T^*} \subset (\ell^1)^* \to (c_0)^*.$$

Notice that the characterization of the subspace D_{T^*} is also required.

(iii) Prove that the operator T is closable. Define the domain $D_{\overline{T}}$ of its closure and determine an element belonging to the set $D_{\overline{T}} \setminus c_c$.

11.4. Compact operators. Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be normed spaces. We denote by

$$K(X, Y) = \{T \in L(X, Y) \mid \overline{T(B_1(0))} \subset Y \text{ compact}\}$$

the set of *compact operators* between X and Y. Prove the following statements.

(i) $T \in L(X,Y)$ is a compact operator if and only if every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in X has a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $(Tx_{n_k})_{k\in\mathbb{N}}$ is convergent in Y.

(ii) If $(Y, \|\cdot\|_Y)$ is complete, then $K(X,Y)$ is a closed subspace of $L(X,Y)$.

(iii) Let $T \in L(X,Y)$. If its range $T(X) \subset Y$ is finite dimensional, then $T \in K(X,Y)$.

(iv) Let $T \in L(X,Y)$ and $S \in L(Y,Z)$. If T or S is a compact operator, then $S \circ T$ is a compact operator.

(v) If X is reflexive, then any operator $T \in L(X,Y)$ which maps weakly convergent sequences to norm-convergent sequences is a compact operator.

11.5. Integral operators ✍. Let $m \in \mathbb{N}$ and let $\Omega \subset \mathbb{R}^m$ be a bounded subset. Given $k \in L^2(\Omega \times \Omega)$, consider the linear operator $K \colon L^2(\Omega) \to L^2(\Omega)$ defined by

$$(Kf)(x) = \int_\Omega k(x,y) f(y)\,\mathrm{d}y$$

(i) Prove that K is well-defined, i.e. $Kf \in L^2(\Omega)$ for any $f \in L^2(\Omega)$.

(ii) Prove that K is a compact operator.

11.6. Operator that is (almost) injective ⧉. Suppose that X, Y, Z are Banach spaces over $\mathbb{R}$, let $P \in L(X,Y)$ and assume that there exists a compact map $J \in L(X,Z)$.

Suppose also that there is a constant $C > 0$ such that for all $x \in X$ one has

$$\|x\|_X \leq C\Big(\|Px\|_Y + \|Jx\|_Z\Big) \tag{$*$}$$

(i) If P is injective, show that there is another constant $C' > 0$ such that for all $x \in X$ one has

$$\|x\|_X \leq C'\|Px\|_Y.$$

(ii) Without assuming that P is injective show that $(*)$ implies that $\ker(P)$ has finite dimension. Hence, prove the existence of a closed subspace W of X with $X = \ker(P) \oplus W$ (i.e. a topologically complementary subspace W of $\ker(P)$ in X). Then exploit part (i) to show that for all $x \in W$ one has

$$\|x\|_X \leq C''\|Px\|_Y$$

for some constant $C'' > 0$.