12.1. Uniform subconvergence. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $C^1([0,1])$ satisfying the following two properties.

$$\begin{aligned} \forall n\in\mathbb{N}: \quad & f_n(0)=f_n'(0),\\ \exists\, C>0 \quad \forall x\in[0,1] \quad \forall n\in\mathbb{N}: \quad & |f_n'(x)|\le C. \end{aligned}$$

Show that $(f_n)_{n\in\mathbb{N}}$ has a uniformly convergent subsequence.

12.2. Sequence with bounded Hölder norm. Let $0<\alpha\le 1$ and let $\Omega\subseteq\mathbb{R}^n$ be an open, bounded subset. For continuous functions $\varphi\colon\Omega\to\mathbb{R}$, consider the so-called *Hölder norm*

$$\|\varphi\|_{C^{0,\alpha}(\Omega)}=\|\varphi\|_{L^\infty(\Omega)}+\sup_{\substack{x,y\in\Omega,\\ x\neq y}}\frac{|\varphi(x)-\varphi(y)|}{|x-y|^\alpha},$$

and the corresponding normed space

$$C^{0,\alpha}(\overline{\Omega},\mathbb{R})=\Big\{\varphi\in C^0(\overline{\Omega},\mathbb{R}) \mid \|\varphi\|_{C^{0,\alpha}(\Omega)}<\infty\Big\}.$$

(i) Prove that a sequence $(\varphi_n)_{n\in\mathbb{N}}\subset C^{0,\alpha}(\overline{\Omega},\mathbb{R})$ which is bounded with respect to the Hölder norm has a uniformly convergent subsequence.

Now let $X=L^2((0,1),\mathbb{R})$ and let $T\colon X\to X$ be given by $T(f)(x)=\int_0^x f(y)\,\mathrm{d}y$.

(ii) Prove that $T(X)\subset C^{0,1/2}((0,1),\mathbb{R})$.

(iii) Use (i) to prove that T is a compact operator from X to $Y=C^0((0,1),\mathbb{R})$.

12.3. Multiplication operators on complex-valued sequences. Let $\ell^p_{\mathbb{C}}$ denote the space of $\mathbb{C}$-valued sequences of summable p-th power, namely

$$\ell^p_{\mathbb{C}}:=\left\{\begin{aligned} x\colon \mathbb{N}&\to\mathbb{C}\\ n&\mapsto x_n\end{aligned} \;\middle|\; \sum_{n\in\mathbb{N}}|x_n|^p<\infty\right\}$$

endowed with its standard Banach norm $\|\cdot\|_{\ell^p_{\mathbb{C}}}$. Given $a\in\ell^\infty_{\mathbb{C}}$ we define the operator $T\colon\ell^2_{\mathbb{C}}\to\ell^2_{\mathbb{C}}$ by $(Tx)_n=a_nx_n$.

(i) Prove that $T\in L(\ell^2_{\mathbb{C}},\ell^2_{\mathbb{C}})$ and compute its operator norm.

(ii) Prove that T is self-adjoint if and only if $a_n\in\mathbb{R}$ for all $n\in\mathbb{N}$.

(iii) Prove that T is compact if and only if $\lim_{n\to\infty}a_n=0$.

12.4. A compact operator on continuous functions. Given $a<b$, let $T\colon C^0([a,b])\to C^0([a,b])$ be the linear operator defined by

$$(Tf)(x)=\int_a^x\frac{f(t)}{\sqrt{x-t}}\,\mathrm{d}t.$$

(i) Check that T is continuous and compute its operator norm $\|T\|$.

(ii) Prove that T is a compact operator.

(iii) Check that the spectral radius r_T of T can be estimated by $r_T \leq 2\sqrt{b-a}$.

12.5. A multiplication operator on square-integrable functions. Given $-\infty < a \leq 0 \leq b < \infty$, let $T\colon L^2([a,b];\mathbb{C}) \to L^2([a,b];\mathbb{C})$ be the linear operator defined by

$$(Tf)(x) = x^2 f(x).$$

(i) Check that T is continuous and compute its operator norm $\|T\|$.

(ii) Prove that T has no eigenvalues.

(iii) Show that T has spectrum $\sigma(T) = [0, \|T\|]$.

12. Solutions

Solution of 12.1: For every $n \in \mathbb{N}$ and $x \in [0,1]$, the assumptions $f_n'(0) = f_n(0)$ and $|f_n'(t)| \leq C$ for all $t \in [0,1]$ imply

$$|f_n(x)| \leq |f_n(0)| + \int_0^x |f_n'(t)|\,\mathrm{d}t = |f_n'(0)| + \int_0^x |f_n'(t)|\,\mathrm{d}t \leq C + x\,C \leq 2C.$$

Consequently, $(f_n)_{n\in\mathbb{N}}$ is uniformly bounded in $C^0([0,1])$. Moreover, it is equicontinuous, since

$$|f_n(x) - f_n(y)| = \left|\int_y^x f_n'(t)\,\mathrm{d}t\right| \leq C|x-y|,$$

hence $|f_n(x) - f_n(y)| < \varepsilon$ whenever $|x-y| < \delta := \frac{\varepsilon}{2C}$. By the Arzelà–Ascoli theorem, $(f_n)_{n\in\mathbb{N}}$ has a uniformly convergent subsequence.

Solution of 12.2:

(i) If $(\varphi_n)_{n\in\mathbb{N}}$ is uniformly bounded in the Hölder norm, then it is uniformly bounded in the sup-norm, since Ω is bounded. Moreover there exists some $C > 0$ so that for every $x, y \in \Omega$ and every $n \in \mathbb{N}$ there holds

$$|\varphi_n(x) - \varphi_n(x)| \leq C|x-y|^\alpha,$$

which implies that the sequence is uniformly continuous. The conclusion then follows by Arzelà-Ascoli's theorem.

(ii) For every $f \in X$, and every $x, y \in (0,1)$ (w.l.o.g. $x \leq y$), by Hölder's inequality we have that

$$|F(f)(x) - F(f)(y)| \leq \int_x^y |f(u)|\,\mathrm{d}u \leq \|f\|_{L^2((0,1))}|x-y|^{1/2},$$

and this yields at once that $F(f) \in C^{0,1/2}((0,1),\mathbb{R})$.

(iii) To prove that F is compact, it suffices to show that, if $(f_n)_{n\in\mathbb{N}}$ is a bounded sequence in X, then up to subsequence $(F(f_n))_{n\in\mathbb{N}}$ converges strongly in $C^0((0,1),\mathbb{R})$ (see Problem 11.4 (i). Similarly as above, for every $n \in \mathbb{N}$ we have

$$|F(f_n)(x) - F(f_n)(y)| \leq C\|f_n\|_{L^2((0,1))}|x-y|^{1/2},$$

and, since the norm $\|f_n\|_{L^2((0,1))}$ is uniformly bounded, it follows that the Hölder 1/2-norm of $(F(f_n))_{n\in\mathbb{N}}$ is uniformly bounded. As a result, thanks to (i), $(F(f_n))_{n\in\mathbb{N}}$ converges strongly in $C^0((0,1),\mathbb{R})$, up to subsequence.

Solution of 12.3:

(i) From

$$\|Tx\|^2_{\ell^2_{\mathbb{C}}} = \sum_{n\in\mathbb{N}} |a_n x_n|^2 \leq \|a\|^2_{\ell^\infty_{\mathbb{C}}} \|x\|^2_{\ell^2_{\mathbb{C}}},$$

we obtain $\|T\| \leq \|a\|_{\ell^\infty_{\mathbb{C}}}$. Given any $k \in \mathbb{N}$, let $e_k = (0, \ldots, 0, 1, 0, \ldots) \in \ell^2_{\mathbb{C}}$, where the 1 is at k-th position. Then, $\|Te_k\|_{\ell^2_{\mathbb{C}}} = |a_k| = |a_k|\|e_k\|_{\ell^2_{\mathbb{C}}}$ implies $\|T\| \geq |a_k|$. Since $k \in \mathbb{N}$ is arbitrary, $\|T\| \geq \|a\|_{\ell^\infty_{\mathbb{C}}}$ follows. To conclude, $\|T\| = \|a\|_{\ell^\infty_{\mathbb{C}}}$.

(ii) The adjoint operator T^* of T is given by $(T^*y)_n = \overline{a_n} y_n$ for $y \in \ell^2_{\mathbb{C}}$ because

$$\forall x, y \in \ell^2_{\mathbb{C}} \qquad (x, T^*y)_{\ell^2_{\mathbb{C}}} = (Tx, y)_{\ell^2_{\mathbb{C}}} = \sum_{n\in\mathbb{N}} a_n x_n \overline{y_n} = \sum_{n\in\mathbb{N}} x_n \overline{\overline{a_n} y_n}.$$

and we conclude $T = T^* \iff a_n = \overline{a_n} \ \ \forall n \in \mathbb{N}$.

(iii) Let $e_k \in \ell^2_{\mathbb{C}}$ be as in (i). Being an orthonormal system of the Hilbert space $\ell^2_{\mathbb{C}}$, the sequence $(e_n)_{n\in\mathbb{N}}$ converges weakly to zero. If T is a compact operator, then $|a_n| = \|Te_n\|_{\ell^2_{\mathbb{C}}} \to 0$ as $n \to \infty$.

Conversely, let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{C}$ such that $a_n \to 0$ as $n \to \infty$ and let $T \in L(\ell^2_{\mathbb{C}}, \ell^2_{\mathbb{C}})$ be the corresponding multiplication operator. Let $(x^{(k)})_{k\in\mathbb{N}}$ be any bounded sequence in $\ell^2_{\mathbb{C}}$ and $C > 0$ a constant such that $\|x^{(k)}\|_{\ell^2_{\mathbb{C}}} \leq C$ for every $k \in \mathbb{N}$. Since $\ell^2_{\mathbb{C}}$ is reflexive, there exists $x \in \ell^2_{\mathbb{C}}$ and an unbounded subset $\Lambda \subset \mathbb{N}$ such that $x^{(k)} \xrightarrow{\text{w}} x$ as $\Lambda \ni k \to \infty$. In particular,

$$\lim_{\Lambda\ni k\to\infty} x_n^{(k)} = \lim_{\Lambda\ni k\to\infty} (e_n, x^{(k)})_{\ell^2_{\mathbb{C}}} = (e_n, x)_{\ell^2_{\mathbb{C}}} = x_n. \qquad (*)$$

Moreover, since $B_C(0; \ell^2_{\mathbb{C}})$ is weakly closed, $\|x\|_{\ell^2_{\mathbb{C}}} \leq C$. Let $\varepsilon > 0$. By assumption, there exists $N \in \mathbb{N}$ such that $|a_n|^2 < \frac{\varepsilon}{4C}$ for all $n \geq N$. Assuming $a \not\equiv 0$, let $K \in \Lambda$ such that for all $\Lambda \ni k \geq K$ and each of the finitely many $n \in \{1, \ldots, N\}$ there holds

$$|x_n^{(k)} - x_n|^2 < \frac{\varepsilon}{2N\|a\|^2_{\ell^\infty_{\mathbb{C}}}}.$$

This is possible due to $(*)$. Then, for all $\Lambda \ni k \geq K$

$$\begin{aligned}\left\|Tx^{(k)} - Tx\right\|^2_{\ell^2_{\mathbb{C}}} &= \sum_{n=1}^{N} |a_n(x_n^{(k)} - x_n)|^2 + \sum_{n=N+1}^{\infty} |a_n(x_n^{(k)} - x_n)|^2 \\ &< \sum_{n=1}^{N} \frac{|a_n|^2\varepsilon}{2N\|a\|^2_{\ell^\infty_{\mathbb{C}}}} + \frac{\varepsilon}{4C} \sum_{n\in\mathbb{N}} \Big(|x_n^{(k)}|^2 + |x_n|^2\Big) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Thus, $(Tx^{(k)})_{k\in\Lambda}$ converges in $\ell^2_{\mathbb{C}}$, which proves that T is a compact operator.

Solution of 12.4:

(i) For every $x \in [a,b]$ and any $f \in C^0([a,b])$ there holds

$$\int_a^x \frac{1}{\sqrt{x-t}}\,\mathrm{d}t = \Big[-2\sqrt{x-t}\Big]_{t=a}^{x} = 2\sqrt{x-a},$$
$$\implies |(Tf)(x)| \leq \int_a^x \frac{|f(t)|}{\sqrt{x-t}}\,\mathrm{d}t \leq 2\sqrt{x-a}\|f\|_{C^0([a,b])}.$$

Therefore, $\|Tf\|_{C^0([a,b])} \leq 2\sqrt{b-a}\|f\|_{C^0([a,b])}$ and $\|T\| \leq 2\sqrt{b-a}$. In fact, choosing a constant function f, we obtain $\|T\| = 2\sqrt{b-a}$.

(ii) Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in $C^0([a,b])$ and let $C>0$ be a constant such that $\|f_n\|_{C^0([a,b])} \leq C$ for all $n \in \mathbb{N}$. Then the sequence $(Tf_n)_{n\in\mathbb{N}}$ is also (uniformly) bounded in $C^0([a,b])$ since

$$\|Tf_n\|_{C^0([a,b])} \leq \|T\|\|f_n\|_{C^0([a,b])} \leq 2C\sqrt{b-a}$$

by part (i). To show equicontinuity, we consider $a \leq x \leq y \leq b$ and estimate

$$\begin{aligned}
|(Tf_n)(y)-(Tf_n)(x)| &= \Big|\int_a^y \frac{f_n(t)}{\sqrt{y-t}}\,\mathrm{d}t - \int_a^x \frac{f_n(t)}{\sqrt{x-t}}\,\mathrm{d}t\Big| \\
&= \Big|\int_x^y \frac{f_n(t)}{\sqrt{y-t}}\,\mathrm{d}t - \int_a^x \Big(\frac{f_n(t)}{\sqrt{x-t}} - \frac{f_n(t)}{\sqrt{y-t}}\Big)\,\mathrm{d}t\Big| \\
&\leq \int_x^y \frac{|f_n(t)|}{\sqrt{y-t}}\,\mathrm{d}t + \int_a^x |f_n(t)|\Big(\frac{1}{\sqrt{x-t}} - \frac{1}{\sqrt{y-t}}\Big)\,\mathrm{d}t \\
&\leq C\bigg(\int_x^y \frac{1}{\sqrt{y-t}}\,\mathrm{d}t + \int_a^x \Big(\frac{1}{\sqrt{x-t}} - \frac{1}{\sqrt{y-t}}\Big)\,\mathrm{d}t\bigg) \\
&\leq 2C\Big(\sqrt{y-x} + \sqrt{x-a} - \sqrt{y-a} + \sqrt{y-x}\Big) \\
&\leq 4C\sqrt{y-x}.
\end{aligned}$$

Hence, $|(Tf_n)(y)-(Tf_n)(x)| < \varepsilon$ whenever $|y-x| < \delta := \frac{\varepsilon^2}{16C^2}$. By the Arzelà–Ascoli theorem, $(Tf_n)_{n\in\mathbb{N}}$ has a uniformly convergent subsequence, which proves that T is a compact operator.

(iii) In part (i) we computed the operator norm $\|T\| = 2\sqrt{b-a}$. By definition,

$$r_T := \inf_{n\in\mathbb{N}} \|T^n\|^{\frac{1}{n}} \leq \|T\| = 2\sqrt{b-a}.$$

Solution of 12.5:

(i) For every $f \in L^2([a,b];\mathbb{C})$, there holds

$$\|Tf\|^2_{L^2([a,b];\mathbb{C})} = \int_a^b x^4|f(x)|^2\,\mathrm{d}x \leq \Big(\max_{x\in[a,b]} x^4\Big)\|f\|^2_{L^2([a,b];\mathbb{C})}$$
$$\implies \|T\| \leq \max\{a^2,b^2\}.$$

Suppose $b > 0$. Let $0 < \varepsilon < b$ and let $f_\varepsilon = \varepsilon^{-\frac{1}{2}}\chi_{[b-\varepsilon,b]}$, where $\chi_{[b-\varepsilon,b]}$ denotes the characteristic function of the interval $[b-\varepsilon, b] \subset [a,b]$. Then,

$$\|Tf_\varepsilon\|^2_{L^2([a,b];\mathbb{C})} = \int_{b-\varepsilon}^{b} x^4 |f_\varepsilon(x)|^2 \,\mathrm{d}x \geq (b-\varepsilon)^4 \|f_\varepsilon\|^2_{L^2([a,b];\mathbb{C})}.$$

Since $\varepsilon > 0$ is arbitrary, we obtain $\|T\| \geq b^2$. Analogously, we can prove $\|T\| \geq a^2$ under the assumption $a < 0$. As a result, we obtain $\|T\| = \max\{a^2, b^2\}$.

(ii) Suppose $\lambda \in \mathbb{C}$ and $f \in L^2([a,b];\mathbb{C})$ satisfy $Tf = \lambda f$. For almost every $x \in [a,b]$,

$$0 = (\lambda f - Tf)(x) = (\lambda - x^2) f(x).$$

From $\lambda - x^2 \neq 0$ for almost all $x \in [a,b]$ we conclude $f(x) = 0$ for almost all $x \in [a,b]$. Hence, $f = 0$ in $L^2([a,b];\mathbb{C})$ which proves that the operator T has no eigenvalues.

(iii) In part (ii) we proved that the operator $(\lambda - T)$ is injective for any $\lambda \in \mathbb{C}$. If the operator $(\lambda - T)$ is surjective for some $\lambda \in \mathbb{C}$, then there exists $f \in L^2([a,b];\mathbb{C})$ with $\lambda f - Tf = 1$. Then, for almost every $x \in [a,b]$,

$$1 = \lambda f(x) - Tf(x) = (\lambda - x^2) f(x) \qquad \Longrightarrow \quad f(x) = \frac{1}{\lambda - x^2}.$$

If $0 \leq \lambda \in \mathbb{R}$ and if $a \leq -\sqrt{\lambda}$ or $\sqrt{\lambda} \leq b$, then $f \notin L^2([a,b])$ in contradiction to our assumption because of the singularity at $x \in [a,b]$ satisfying $x^2 = \lambda$. Therefore, $(\lambda - T)$ is not surjective for $\lambda \in [0, \max\{a^2, b^2\}]$ which shows $[0, \|T\|] \subset \sigma(T)$.

If $\lambda \in \mathbb{C}\setminus[0, \|T\|]$, then the function $f\colon [a,b] \to \mathbb{C}$ with $f(x) = \frac{1}{\lambda - x^2}$ is bounded. Therefore, the map $R_\lambda\colon L^2([a,b];\mathbb{C}) \to L^2([a,b];\mathbb{C})$ given by $g \mapsto gf$ is continuous. Moreover, by construction $(\lambda - T)(gf) = g$ for any $g \in L^2([a,b];\mathbb{C})$, which proves $(\lambda - T)^{-1} = R_\lambda$. To conclude, $\sigma(T) = [0, \|T\|]$.