13.1. Definitions of resolvent set. Let $(X, \|\cdot\|_X)$ be a Banach space over $\mathbb{C}$ and let $A\colon D_A \subset X \to X$ be a linear operator. Prove that if A has closed graph, then the following sets coincide.

$$\begin{aligned}\rho(A) &= \{\lambda \in \mathbb{C} \mid (\lambda - A)\colon D_A \to X \text{ is bijective, } \exists(\lambda - A)^{-1} \in L(X, X)\},\\ \tilde{\rho}(A) &= \{\lambda \in \mathbb{C} \mid (\lambda - A)\colon D_A \to X \text{ is injective with dense image,}\\ &\qquad\qquad\qquad\qquad\qquad\qquad \exists(\lambda - A)^{-1} \in L(Z(\lambda), X)\},\end{aligned}$$

where we have set $Z(\lambda) := (\lambda - A)(D_A)$, i.e., the image of $\lambda - A$, and $(\lambda - A)^{-1} \in L(Z(\lambda), X)$ means that the (necessarily linear) set-theoretic inverse of $\lambda - A$ is bounded, in the usual sense that $\sup_{z \in Z_\lambda, \|z\|_X \leq 1} \|(\lambda - A)^{-1}(z)\| < \infty$.

Remark. In the literature, the resolvent set is often defined to be $\tilde{\rho}(A)$ rather than $\rho(A)$. Since as soon as $\rho(A)$ is not empty A has closed graph, this problem shows that the two perspectives are in fact equivalent.

13.2. Unitary operators.

Definition. Let $(H, \langle\cdot,\cdot\rangle_H)$ be a Hilbert space over $\mathbb{C}$. An invertible linear operator $T \in L(H, H)$ is called *unitary*, if $T^* = T^{-1}$.

(i) Prove that $T \in L(H, H)$ is unitary if and only if T is a bijective isometry.

(ii) Prove that if $T \in L(H, H)$ is unitary, then its spectrum lies on the unit circle:

$$\sigma(T) \subset \mathbb{S}^1 := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}.$$

13.3. Integral operators revisited. Let $\Omega \subset \mathbb{R}^m$ be a bounded subset. Given $k \in L^2(\Omega \times \Omega)$ such that $k(x, y) = k(y, x)$ for almost every $(x, y) \in \Omega \times \Omega$, consider the operator $K\colon L^2(\Omega) \to L^2(\Omega)$ defined by

$$(Kf)(x) = \int_\Omega k(x, y) f(y)\, \mathrm{d}y$$

and the operator

$$\begin{aligned} A\colon L^2(\Omega) &\to L^2(\Omega)\\ f &\mapsto f - Kf. \end{aligned}$$

Prove that injectivity of A and surjectivity of A are equivalent.

13.4. Resolvents and spectral distance. Let $(H, \langle\cdot,\cdot\rangle_H)$ be a Hilbert space over $\mathbb{C}$.

(i) Let $A \in L(H, H)$ be a self-adjoint operator and let $\lambda \in \rho(A)$ be an element in its resolvent set. Show that the resolvent $R_\lambda := (\lambda - A)^{-1}$ is a *normal* operator, i.e.,

$$R_\lambda R_\lambda^* = R_\lambda^* R_\lambda.$$

(ii) Let $A, B \in L(H, H)$ be self-adjoint operators. The *Hausdorff distance* of their spectra $\sigma(A), \sigma(B) \subset \mathbb{C}$ is defined to be

$$d\Big(\sigma(A), \sigma(B)\Big) := \max\left\{ \sup_{\alpha\in\sigma(A)} \Big(\inf_{\beta\in\sigma(B)} |\alpha-\beta| \Big), \sup_{\beta\in\sigma(B)} \Big(\inf_{\alpha\in\sigma(A)} |\alpha-\beta| \Big) \right\}.$$

Prove the estimate

$$d\Big(\sigma(A), \sigma(B)\Big) \leq \|A - B\|_{L(H,H)}.$$

Remark. The Hausdorff distance d is in fact a distance on compact subsets of $\mathbb{C}$. In particular, it restricts to an actual distance function on the spectra of bounded linear operators.

13.5. Compact operator on space decomposition. Let H be a Hilbert space over $\mathbb{R}$ and let $A\colon H \to H$ be linear, compact and self-adjoint.

(i) State the spectral theorem for A.

Now, suppose the existence of two complementary and mutually orthogonal subspaces $H', H'' \subset H$ that are A-invariant, meaning that

$$H = H' \oplus^{\perp} H'', \qquad A(H') \subset H', \qquad A(H'') \subset H''.$$

(ii) Show that each of the restricted operators $A' := A_{|H'}$ and $A'' := A_{|H''}$ is also compact and self-adjoint.

Assume now that A is nonnegative definite (i.e., $(Ax, x) \geq 0$ for all $x \in H$).

(iii) State the Courant–Fischer characterization of the eigenvalues of A.

(iv) Denoted by $\lambda_1, \lambda_1', \lambda_1''$ the first (namely, the *largest*) eigenvalue of A, A', A'' respectively, show that

$$\lambda_1 = \max\{\lambda_1', \lambda_1''\}.$$

13.6. Heisenberg's uncertainty principle. Let $(H, \langle\cdot,\cdot\rangle_H)$ be a Hilbert space over $\mathbb{C}$. Let $D_A, D_B \subset H$ be dense subspaces and let $A\colon D_A \subset H \to H$ and $B\colon D_B \subset H \to H$ be symmetric linear operators. Under the necessary assumption that $A(D_A \cap D_B) \subset D_B$ and $B(D_A \cap D_B) \subset D_A$, the *commutator*

$$\begin{aligned} [A, B]\colon D_{[A,B]} \subset H &\to H \\ x &\mapsto A(Bx) - B(Ax) \end{aligned}$$

is a well-defined operator on $D_{[A,B]} := D_A \cap D_B$.

(i) Prove the following inequality:

$$\forall x \in D_{[A,B]}: \qquad 2\|Ax\|_H \|Bx\|_H \geq \Big|\langle x, [A, B]x\rangle_H\Big|.$$

(ii) Given the symmetric operator $A\colon D_A \subset H \to H$ we define the *standard deviation*

$$\varsigma(A,x) := \sqrt{\langle Ax, Ax\rangle_H - \langle x, Ax\rangle_H^2}$$

at each $x \in D_A$ with $\|x\|_H = 1$. Verify $\varsigma(A,x) \in \mathbb{R}$ and prove the following inequality:

$$\forall x \in D_{[A,B]},\ \|x\|_H = 1: \quad 2\varsigma(A,x)\,\varsigma(B,x) \geq \Big|\langle x, [A,B]x\rangle_H\Big|.$$

Remark. The possible *states* of a quantum mechanical system are given by elements $x \in H$ with $\|x\|_H = 1$. Each *observable* is given by a symmetric linear operator $A\colon D_A \subset H \to H$. If the system is in state $x \in D_A$, we measure the observable A with uncertainty $\varsigma(A,x)$.

(iii) Let $A\colon D_A \to H$ and $B\colon D_B \to H$ be as above. The pair of operators (A,B) is called *Heisenberg pair* if

$$[A,B] = i\,\mathrm{id}_{D_{[A,B]}}\,.$$

Under the assumption that B has finite operator norm and $D_B = H$, prove that if (A,B) is a Heisenberg pair, then $A\colon D_A \subset H \to H$ cannot have finite operator norm.

(iv) Consider the Hilbert space $(H, \langle\cdot,\cdot\rangle_H) = \Big(L^2([0,1];\mathbb{C}), \langle\cdot,\cdot\rangle_{L^2}\Big)$ and the subspace

$$C^1_0([0,1];\mathbb{C}) := \{f \in L^2([0,1];\mathbb{C}) \mid f \in C^1([0,1];\mathbb{C}),\ f(0) = 0 = f(1)\}.$$

Here, we denote elements in the Hilbert space $L^2([0,1];\mathbb{C})$ by f and points in the interval $[0,1]$ by s. We understand $f \in C^1([0,1];\mathbb{C})$ if f has a representative in C^1 and write $f' = \frac{\mathrm{d}}{\mathrm{d}s} f$ in this case. Recall that in this sense, $C^1_0([0,1];\mathbb{C}) \subset L^2([0,1];\mathbb{C})$ is a dense subspace. The operators

$$P\colon C^1_0([0,1];\mathbb{C}) \to L^2([0,1];\mathbb{C}), \quad f(s) \mapsto if'(s) \qquad\qquad Q\colon L^2([0,1];\mathbb{C}) \to L^2([0,1];\mathbb{C}), \quad f(s) \mapsto sf(s)$$

correspond to the observables *momentum* and *position*. Check that P and Q are well-defined, symmetric operators. Check that $[P,Q]\colon C^1_0([0,1];\mathbb{C}) \to L^2([0,1];\mathbb{C})$ is well-defined.

Show that (P,Q) is a Heisenberg pair and conclude the uncertainty principle:

$$\forall f \in C^1_0([0,1];\mathbb{C}),\ \|f\|_{L^2([0,1];\mathbb{C})} = 1: \quad \varsigma(P,f)\,\varsigma(Q,f) \geq \tfrac{1}{2}.$$

The more precisely the momentum of some particle is known, the less precisely its position can be known, and vice versa.