1.1. Equivalent Norms ⚙.

Definition. Let X be a set. A *metric* on X is a non-negative function $d\colon X \times X \to \mathbb{R}$ that satisfies for all $x, y, z \in X$

$$d(x,y) = 0 \Leftrightarrow x = y, \qquad d(x,y) = d(y,x), \qquad d(x,z) \leq d(x,y) + d(y,z).$$

We say that two metrics d and d' on X are *equivalent* if

$$\exists C > 0 \quad \forall x_1, x_2 \in X: \quad C^{-1} d'(x_1,x_2) \leq d(x_1,x_2) \leq C d'(x_1,x_2).$$

Let X be a vector space over $\mathbb{R}$. A *norm* on X is a non-negative function $\|\cdot\|\colon X \to \mathbb{R}$ which satisfies for all $x, y \in X$ and $\lambda \in \mathbb{R}$

$$\|x\| = 0 \Leftrightarrow x = 0, \qquad \|\lambda x\| = |\lambda| \|x\|, \qquad \|x + y\| \leq \|x\| + \|y\|.$$

We say that two norms $\|\cdot\|$ and $\|\cdot\|'$ on X are *equivalent* if

$$\exists C > 0 \quad \forall x \in X: \quad C^{-1} \|x\|' \leq \|x\| \leq C \|x\|'.$$

Recall that a norm $\|\cdot\|$ induces a metric $d_{\|\cdot\|}$ by the formula $d_{\|\cdot\|}(x_1, x_2) = \|x_1 - x_2\|$.

(i) Let X be a finite-dimensional vector space over $\mathbb{R}$. Show that all norms on X are equivalent.

(ii) Construct two metrics on $\mathbb{R}^2$ that are *not* equivalent.

(iii) Construct a vector space X with two norms $\|\cdot\|$ and $\|\cdot\|'$ that are *not* equivalent.

Hint. Prove that $\|\cdot\|$ and $\|\cdot\|'$ are not equivalent by exhibiting a sequence $(x_n) \subset X$ that converges for $\|\cdot\|$ but not for $\|\cdot\|'$.

1.2. Intrinsic Characterisations ⚙.

Let V be a vector space over $\mathbb{R}$. Prove the following equivalences.

(i) The norm $\|\cdot\|$ is induced by a scalar product $\langle \cdot, \cdot \rangle$ (in the sense that there exists a scalar product $\langle \cdot, \cdot \rangle$ such that $\forall x \in V: \ \|x\|^2 = \langle x, x \rangle$)

$\Leftrightarrow$ the norm satisfies the *parallelogram identity*, i.e., $\forall x, y \in V$:

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Hint. If $\|\cdot\|$ satisfies the parallelogram identity, consider $\langle x, y \rangle := \frac{1}{4}\|x + y\|^2 - \frac{1}{4}\|x - y\|^2$. Prove $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$ first for $\lambda \in \mathbb{N}$, then for $\lambda \in \mathbb{Q}$ and finally for $\lambda \in \mathbb{R}$.

(ii) The metric $d(\cdot, \cdot)$ is induced by a norm $\|\cdot\|$ (in the sense that there exists a norm $\|\cdot\|$ such that $\forall x, y \in V: \ d(x,y) = \|x - y\|$)

$\Leftrightarrow$ the metric is *translation invariant* and *homogeneous*, i.e., $\forall v, x, y \in V \ \forall \lambda \in \mathbb{R}$:

$$d(x + v, y + v) = d(x,y),$$
$$d(\lambda x, \lambda y) = |\lambda| d(x,y).$$

1.3. When $L^p(\mathbb{R})$ and $\ell^p(\mathbb{N})$ are Hilbert spaces.

(i) Determine all values of $p \in [1,\infty]$ such that the Banach space $L^p(\mathbb{R})$ is actually Hilbertean (meaning that the norm $\|\cdot\|_{L^p}$ is induced by a scalar product).

(ii) Determine all values of $p \in [1,\infty]$ such that the Banach space $\ell^p(\mathbb{N})$ is actually Hilbertean (meaning that the norm $\|\cdot\|_{\ell^p}$ is induced by a scalar product).

It is advised not to forget the case $p=\infty$ in your discussion.

1.4. When a distance is induced by a norm.

(i) Let V be a vector space over $\mathbb{R}$ and let $d\colon V\times V\to\mathbb{R}$ be a distance. State necessary and sufficient conditions for $d(\cdot,\cdot)$ to be induced by a norm $\|\cdot\|$, in the sense that

$$d(v_1,v_2)=\|v_1-v_2\| \quad \forall\, v_1,v_2\in V.$$

(Note that only a statement is requested, no proof.)

(ii) Consider the vector space $C([0,\infty);\mathbb{R})$ consisting of continuous functions defined on $[0,\infty)\subset\mathbb{R}$ and attaining real values, and the distance

$$d(f_1,f_2)=\sum_{n=1}^{\infty}2^{-n}\frac{\|f_1-f_2\|_{C^0([0,n])}}{1+\|f_1-f_2\|_{C^0([0,n])}}$$

where $\|f\|_{C^0([0,n])}=\sup_{x\in[0,n]}|f(x)|$. Is d induced by a norm?

1.5. Infinite-dimensional vector spaces and separability.

(i) Let $\emptyset\neq\Omega\subset\mathbb{R}^n$ be an open set. Show that $L^p(\Omega)$ is an infinite-dimensional vector space for all $1\leq p\leq\infty$.

(ii) Let $(X,\mathcal{A},\mu)$ be a measure space. Recall that if X is separable and the measure μ is finite (or, more generally, σ-finite) and if $1\leq p<\infty$, then the space $L^p(X,\mathcal{A},\mu)$ is separable. Roughly speaking, in the simple case when $X=(0,1)$, $\mathcal{A}=$ Borel-σ-algebra and $\mu=\mathscr{L}^1$, this relies on the fact that any element in those spaces can be arbitrarily well approximated by a function of the form

$$f=\sum_{i=1}^{k}q_i\,\chi_{B_i}\qquad\text{for } k\in\mathbb{N},\ B_i:=B_{r_i}(x_i),\ q_i\in\mathbb{Q},\ x_i\in\mathbb{Q}\cap(0,1),\ 0<r_i\in\mathbb{Q}.$$

Show that instead $\left(L^\infty((0,1)),\|\cdot\|_{L^\infty((0,1))}\right)$ is *not* separable, i.e., it does not contain a countable dense subset.

(Recall that $\|u\|_{L^\infty((0,1))}:=\inf\{K>0\mid |u(x)|\leq K \text{ for almost every } x\in(0,1)\}$.)

1. Solutions

Solution of 1.1:

(i) Let $n = \dim X$ and let $\{e_1, \dots, e_n\}$ be a basis for X. Then every $x \in X$ is of the form $x = \sum_{k=1}^n x_k e_k$ with uniquely determined components $x_1, \dots, x_n \in \mathbb{R}$. Recall that

$$\|x\|_\infty := \max_{k \in \{1,\dots,n\}} |x_k|$$

defines a norm on X. We show that any given norm $\|\cdot\|$ is equivalent to $\|\cdot\|_\infty$ and therefore any two norms are equivalent to each other. We have

$$\begin{aligned} \|x\| = \Big\| \sum_{k=1}^n x_k e_k \Big\| &\leq \sum_{k=1}^n \|x_k e_k\| = \sum_{k=1}^n |x_k| \|e_k\| \\ &\leq n \Big(\max_{k \in \{1,\dots,n\}} |x_k| \Big) \Big(\max_{k \in \{1,\dots,n\}} \|e_k\| \Big) = nM\|x\|_\infty \end{aligned} \tag{$*$}$$

where

$$M := \Big(\max_{k \in \{1,\dots,n\}} \|e_k\| \Big)$$

is a finite constant. The triangle inequality implies $\big| \|x\| - \|y\| \big| \leq \|x - y\|$. Combined with $(*)$ we have

$$\big| \|x\| - \|y\| \big| \leq \|x - y\| \leq nM\|x - y\|_\infty$$

for every $x, y \in X$. This implies that $\|\cdot\| \colon (X, \|\cdot\|_\infty) \to \mathbb{R}$ is a continuous map. We restrict this map to $K := \{x \in X : \|x\|_\infty = 1\}$. Note that K is a closed and bounded subset of $(X, \|\cdot\|_\infty)$. Moreover recall that $\|\cdot\|_\infty$ is equivalent to the Euclidean norm $\|\cdot\|_2$, which is defined as $\|x\|_2 := (\sum_{k=1}^n |x_k|^2)^{1/2}$ for every $x = \sum_{k=1}^n x_k e_k \in X$ (in particular $\|x\|_\infty \leq \|x\|_2 \leq n\|x\|_\infty$). Hence, by Heine-Borel theorem, K is compact. Therefore, the function $\|\cdot\|$ attains minimum and maximum on K, i.e., there exists $x_1, x_2 \in X$ such that

$$m_1 := \min_{x \in K} \|x\| = \|x_1\|, \qquad\qquad m_2 := \max_{x \in K} \|x\| = \|x_2\|.$$

Since $\|x_1\|_\infty = 1$ we have $x_1 \neq 0$ and $m_1 > 0$. Then, for an arbitrary $x \in X \setminus \{0\}$ we have

$$\frac{x}{\|x\|_\infty} \in K \quad \Longrightarrow \quad 0 < m_1 \leq \Big\| \frac{x}{\|x\|_\infty} \Big\| \leq m_2 < \infty.$$

Multiplication with $\|x\|_\infty$ implies

$$0 < m_1\|x\|_\infty \leq \|x\| \leq m_2\|x\|_\infty < \infty.$$

Any other given norm $\|\cdot\|'$ satisfies analogously

$$0 < m_1'\|x\|_\infty \leq \|x\|' \leq m_2'\|x\|_\infty < \infty.$$

Thus, the combination of the two last inequalities proves the equivalence of $\|\cdot\|$ and $\|\cdot\|'$.

(ii) Let d be the metric on $\mathbb{R}^2$ induced by the Euclidean norm. We define d' on $\mathbb{R}^2$ by

$$d'(x,y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}$$

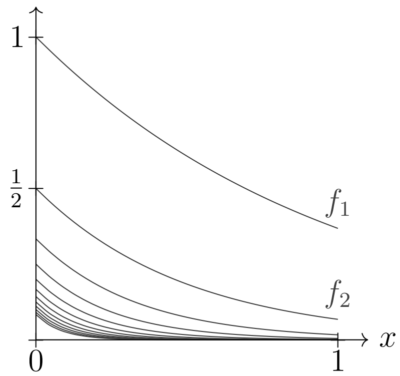
Let z be a point on the Euclidean unit circle $\mathbb{S}^1 \subset \mathbb{R}^2$ let $z_n = \frac{1}{n}z$. Then, $d(0, z_n) = \frac{1}{n}$ and $d'(0, z_n) = 1$. Since an inequality of the form $1 \leq C\frac{1}{n}$ cannot hold for every $n \in \mathbb{N}$ if C is finite, d and d' are not equivalent.

(iii) Let $X = C^1([0,1])$. Let $\|\cdot\|$ and $\|\cdot\|'$ be the two norms on X given by

$$\|u\| := \|u\|_{C^0} = \sup_{x\in[0,1]} |u(x)|, \qquad \|u\|' := \max\left\{\sup_{x\in[0,1]} |u(x)|, \sup_{x\in[0,1]} |u'(x)|\right\}$$

For $n \in \mathbb{N}$ we consider

$$f_n \colon [0,1] \to \mathbb{R}$$
$$x \mapsto \frac{e^{-nx}}{n}.$$



Then, $f_n \in C^1([0,1])$ for every $n \in \mathbb{N}$. Moreover, $\|f_n\| = \frac{1}{n}$ and $\|f_n\|' = 1$. Since an inequality of the form $1 \leq C\frac{1}{n}$ cannot hold for every $n \in \mathbb{N}$ if C is finite, $\|\cdot\|$ and $\|\cdot\|'$ are not equivalent.

Solution of 1.2:

(i) If the norm $\|\cdot\|$ is induced by the scalar product $\langle\cdot,\cdot\rangle$, then the parallelogram identity holds:

$$\begin{aligned}
&\|x+y\|^2 + \|x-y\|^2 \\
&= \langle x+y, x+y\rangle + \langle x-y, x-y\rangle \\
&= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle + \langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle \\
&= 2\|x\|^2 + 2\|y\|^2.
\end{aligned}$$

Conversely, we assume that $\|\cdot\|$ satisfies the parallelogram identity and claim that

$$\langle x,y\rangle := \tfrac{1}{4}\|x+y\|^2 - \tfrac{1}{4}\|x-y\|^2$$

defines a scalar product inducing $\|\cdot\|$.

- *Symmetry.* Since $\|x-y\| = \|(-1)(y-x)\| = \|y-x\|$ and since $x+y = y+x$, we have $\langle x,y\rangle = \langle y,x\rangle$ for all $x,y \in V$.

- *Linearity.* Let $x,y,z \in V$. We use the parallelogram identity in the following way.

$$\|(x+z)+y\|^2 + \|(x+z)-y\|^2 = 2\|x+z\|^2 + 2\|y\|^2.$$

We rewrite the equation above to obtain

$$\|x+y+z\|^2 = 2\|x+z\|^2 + 2\|y\|^2 - \|x-y+z\|^2 =: A$$

and switch the roles of x and y to get

$$\|x+y+z\|^2 = 2\|y+z\|^2 + 2\|x\|^2 - \|y-x+z\|^2 =: B.$$

Therefore,

$$\begin{aligned}\|x+y+z\|^2 &= \frac{A}{2} + \frac{B}{2} \\ &= \|x+z\|^2 + \|y\|^2 + \|y+z\|^2 + \|x\|^2 - \frac{\|x-y+z\|^2 + \|y-x+z\|^2}{2}. \qquad (1)\end{aligned}$$

Analogously,

$$\begin{aligned}&\|x+y-z\|^2 \\ &= \|x-z\|^2 + \|y\|^2 + \|y-z\|^2 + \|x\|^2 - \frac{\|x-y-z\|^2 + \|y-x-z\|^2}{2}. \qquad (2)\end{aligned}$$

Note that the last term of (1) agrees with the last term of (2). Hence, we have

$$\begin{aligned}\langle x+y, z\rangle &= \tfrac{1}{4}\|x+y+z\|^2 - \tfrac{1}{4}\|x+y-z\|^2 \\ &= \tfrac{1}{4}\Big(\|x+z\|^2 + \|y+z\|^2 - \|x-z\|^2 - \|y-z\|^2\Big) = \langle x, z\rangle + \langle y, z\rangle.\end{aligned}$$

Given $n \in \mathbb{N}$, by the additivity that we just proved, we have

$$\langle nx, z\rangle = \Big\langle \sum_{k=1}^{n} x, z \Big\rangle = \sum_{k=1}^{n} \langle x, z\rangle = n\langle x, z\rangle$$

Moreover, since $\langle 0, y\rangle = \frac{1}{4}\Big(\|y\|^2 - \|y\|^2\Big) = 0$,

$$0 = \langle 0, y\rangle = \langle x - x, y\rangle = \langle x, y\rangle + \langle -x, y\rangle \qquad \Rightarrow \ \langle -x, y\rangle = -\langle x, y\rangle.$$

Consequently, $\langle mx, z\rangle = m\langle x, z\rangle$ for every $m \in \mathbb{Z}$. Now, given $m \in \mathbb{Z}$ and $n \in \mathbb{N}^*$, it holds

$$\Big\langle \frac{m}{n}x, z \Big\rangle = \frac{n}{n} m \Big\langle \frac{1}{n}x, z \Big\rangle = \frac{m}{n}\Big\langle \frac{n}{n}x, z \Big\rangle = \frac{m}{n}\langle x, z\rangle,$$

which implies $\langle qx, z\rangle = q\langle x, z\rangle$ for every $q \in \mathbb{Q}$.

Let $\lambda \in \mathbb{R}$ and let $(q_n)_{n\in\mathbb{N}}$ be a sequence of rational numbers converging to λ for $n \to \infty$. Since the triangle inequality $|\|x\| - \|y\|| \leq \|x - y\|$ implies that the norm is a continuous map, we have

$$\begin{aligned}\langle \lambda x, z\rangle = \tfrac{1}{4}\|\lambda x + z\|^2 - \tfrac{1}{4}\|\lambda x - z\|^2 &= \lim_{n\to\infty}\Big(\tfrac{1}{4}\|q_n x + z\|^2 - \tfrac{1}{4}\|q_n x - z\|^2\Big) \\ &= \lim_{n\to\infty}\langle q_n x, z\rangle = \lim_{n\to\infty} q_n\langle x, z\rangle = \lambda\langle x, z\rangle.\end{aligned}$$

Linearity in the second argument follows by symmetry.

- *Positive-definiteness.* For all $x \in V$, we have

$$\langle x, x\rangle = \tfrac{1}{4}\|x+x\|^2 - \tfrac{1}{4}\|x-x\|^2 = \tfrac{1}{4}\|2x\|^2 = \|x\|^2 \geq 0.$$

This also shows that $\|\cdot\|$ is induced by $\langle\cdot,\cdot\rangle$. Moreover, $\langle x, x\rangle = 0 \Leftrightarrow \|x\| = 0 \Leftrightarrow x = 0$.

(ii) If the metric d is induced by the norm $\|\cdot\|$, then

$$d(x+v, y+v) = \|(x+v)-(y+v)\| = \|x-y\| = d(x,y),$$
$$d(\lambda x, \lambda y) = \|\lambda x - \lambda y\| = \|\lambda(x-y)\| = |\lambda|\|x-y\|.$$

Conversely, we assume that the metric d is translation invariant and homogeneous and claim that

$$\|x\| := d(x,0)$$

defines a norm inducing d. The function $\|\cdot\|$ is indeed a norm, because for all $x,y,z \in V$ and $\lambda \in \mathbb{R}$, we have

$$\|x\| = 0 \;\Leftrightarrow\; d(x,0) = 0 \;\Leftrightarrow\; x = 0,$$
$$\|\lambda x\| = d(\lambda x, 0) = d(\lambda x, \lambda 0) = |\lambda| d(x,0) = |\lambda|\|x\|,$$
$$\|x+y\| = d(x+y,0) \leq d(x+y,y) + d(y,0) = d(x,0) + d(y,0) = \|x\| + \|y\|.$$

Moreover, $\|\cdot\|$ induces the metric d since for all $x,y \in V$

$$\|x-y\| = d(x-y,0) = d(x,y).$$

Solution of 1.3: The parallelogram identity reads

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

and it is a necessary and sufficient condition for a Banach space to be Hilbertian, as seen in point (i) of Problem 1.2.

(i) For any $1 \leq p \leq \infty$, we can consider the characteristic functions $\chi_{[0,1]} \in L^p(\mathbb{R})$ and $\chi_{[1,2]} \in L^p(\mathbb{R})$. Then,

$$2\|\chi_{[0,1]}\|^2_{L^p(\mathbb{R})} = 2\|\chi_{[1,2]}\|^2_{L^p(\mathbb{R})} = 2$$
$$\|\chi_{[0,1]} + \chi_{[1,2]}\|^2_{L^p(\mathbb{R})} = \|\chi_{[0,1]} - \chi_{[1,2]}\|^2_{L^p(\mathbb{R})} = \begin{cases} 1 & \text{if } p = \infty, \\ 2^{\frac{2}{p}} & \text{else.} \end{cases}$$

Hence, the parallelogram identity is violated for $p = \infty$, while for $1 \leq p < \infty$ it is fulfilled if and only if $2^{\frac{2}{p}} + 2^{\frac{2}{p}} = 2 + 2$, which is true if and only if $p = 2$.

Hence $L^p(\mathbb{R})$ is Hilbertean if and only if $p = 2$. In fact $L^2(\mathbb{R})$ is a Hilbert space with respect to the scalar product

$$\langle f, g \rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} f(x) g(x) \, \mathrm{d}x.$$

(ii) For any $1 \leq p \leq \infty$, we can consider the elements $x = (1, 0, 0, \dots) \in \ell^p(\mathbb{N})$ and $y = (0, 1, 0, \dots) \in \ell^p(\mathbb{N})$. Then,

$$\begin{aligned} 2\|x\|_{\ell^p}^2 &= 2\|y\|_{\ell^p}^2 = 2 \\ \|x+y\|_{\ell^p}^2 &= \|x-y\|_{\ell^p}^2 = \begin{cases} 1 & \text{if } p = \infty, \\ 2^{\frac{2}{p}} & \text{else.} \end{cases} \end{aligned}$$

Hence, as in the previous point, the parallelogram identity is violated for all $p \neq 2$. For $p = 2$, the space $\ell^2(\mathbb{R})$ is indeed Hilbertean by virtue of the scalar product

$$\langle x, y \rangle_{\ell^2} = \sum_{n \in \mathbb{N}} x_n y_n.$$

Solution of 1.4:

(i) The distance $d(\cdot, \cdot)$ on the vector space V is induced by a norm if and only if

$$\begin{aligned} \forall x, y, v \in V: \qquad & d(x+v, y+v) = d(x, y), \\ \forall x, y \in V \quad \forall \lambda \in \mathbb{R}: \qquad & d(\lambda x, \lambda y) = |\lambda| d(x, y) \end{aligned}$$

(see (ii) in Problem 1.2).

(ii) Let $f \in C^0([0, \infty)) \setminus \{0\}$ be supported in $[0, 1]$ and let $\lambda > 0$. Then

$$d(\lambda f, 0) = \Big(\sum_{n=1}^{\infty} 2^{-n} \Big) \frac{\lambda \|f\|_{C^0([0,1])}}{1 + \lambda \|f\|_{C^0([0,1])}} \xrightarrow{\lambda \to \infty} 1$$

which proves that d is not homogeneous and thus not induced by a norm.

Solution of 1.5:

(i) Suppose by contradiction that $L^p(\Omega)$ has finite dimension $d \in \mathbb{N}$. Since $\emptyset \neq \Omega \subset \mathbb{R}^n$ is open, there exist $d+1$ disjoint balls $B_i := B_{r_i}(x_i) \subset \Omega$ for $i = 1, \dots, d+1$. Let $\varphi_i(x) := \chi_{B_i}$ be the characteristic function of B_i. Then, $\varphi_1, \dots, \varphi_{d+1} \in L^p(\Omega)$ have disjoint supports. Moreover, since the subset $\{\varphi_1, \dots, \varphi_{d+1}\}$ contains more than d elements, it must be linearly dependent. Let $\lambda_1, \dots, \lambda_{d+1} \in \mathbb{R}$ be not all equal to 0 such that

$$\sum_{i=1}^{d+1} \lambda_i \varphi_i = 0.$$

However, if we multiply by φ_j for any $j \in \{1, \dots, d+1\}$ and integrate over Ω, we get

$$0 = \int_\Omega \sum_{i=1}^{d+1} \lambda_i \varphi_i \varphi_j \, \mathrm{d}\mu = \int_\Omega \lambda_j \varphi_j^2 \, \mathrm{d}\mu = \lambda_j \int_\Omega \varphi_j^2 \, \mathrm{d}\mu \qquad \Rightarrow \lambda_j = 0,$$

which contradicts the fact that $L^p(\Omega)$ is finite dimensional.

(ii) We define $I_n := (\frac{1}{n+1}, \frac{1}{n}) \subset (0,1)$ for $n \in \mathbb{N}$ and consider the characteristic function χ_{I_n} of I_n, i.e.,

$$\chi_{I_n}(x) := \begin{cases} 1, & \text{if } x \in I_n, \\ 0, & \text{if } x \in (0,1) \setminus I_n. \end{cases}$$

Given any subset $\emptyset \neq M \subset \mathbb{N}$ we define the function $f_M \in L^\infty((0,1))$ by

$$f_M(x) := \sum_{n \in M} \chi_{I_n}(x)$$

Since the intervals I_n are pairwise disjoint, open and non-empty, we have $\|f_M\|_{L^\infty} = 1$ for every $\emptyset \neq M \subset \mathbb{N}$. For the same reason,

$$\|f_M - f_{M'}\|_{L^\infty} = 1,$$

if $M \neq M'$. Therefore, the balls $B_M = \{g \in L^\infty((0,1)) \mid \|g - f_M\|_{L^\infty} < \frac{1}{3}\}$ are pairwise disjoint. If $S \subset L^\infty((0,1))$ is any dense subset, then $S \cap B_M \neq \emptyset$ for every $\emptyset \neq M \subset \mathbb{N}$. Thus, there is a surjective map $S \to \{B_M \mid \emptyset \neq M \subset \mathbb{N}\}$. Since there are uncountably many different subsets of $\mathbb{N}$, the set S must be uncountable as well. Therefore, $L^\infty((0,1))$ does not admit a countable dense subset.