2.1. Statements of Baire ✍.

Definition. Let (M, d) be a metric space and consider a subset $A \subset M$. Then, $\overline{A}$ denotes the closure, A° the interior and $A^\complement = M \setminus A$ the complement of A. We say that A is

- *dense*, if $\overline{A} = X$;
- *nowhere dense*, if $(\overline{A})^\circ = \emptyset$;
- *meagre*, if $A = \bigcup_{n\in\mathbb{N}} A_n$ is a countable union of nowhere dense sets A_n;
- *residual*, if $A^\complement$ is meagre.

Show that the following statements are equivalent.

(i) Every residual set $\Omega \subset M$ is dense in M.

(ii) The interior of every meagre set $A \subset M$ is empty.

(iii) The empty set is the only subset of M that is open and meagre.

(iv) Countable intersections of dense open sets are dense.

Hint. Show (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i). Use that subsets of meagre sets are meagre and recall that $A \subset M$ is dense $\Leftrightarrow$ $\overline{A} = M$ $\Leftrightarrow$ $(M \setminus A)^\circ = \emptyset$.

Remark. Baire's theorem states that (i), (ii), (iii), (iv) are true if (M, d) is complete.

2.2. Quick warm-up: true or false? ✍. Decide whether the following statements are true or false. If true, think of a quick proof. If false, find a simple counterexample. *(self-check: not to be handed in.)*

(i) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions $f_n \in C^0([0,1])$. If there exists $f\colon [0,1] \to \mathbb{R}$ such that $\forall x \in [0,1] : \lim_{n\to\infty} f_n(x) = f(x)$ then $f \in C^0([0,1])$.

(ii) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions $f_n \in C^0([0,1])$. If $\forall x \in [0,1]$ $\exists\, C(x) : \sup_{n\in\mathbb{N}} |f_n(x)| \leq C(x)$ then $\sup_{n\in\mathbb{N}} \sup_{x\in[0,1]} |f_n(x)| < \infty$.

(iii) The function $d\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ given by $d(x, y) = \min\{|x_1 - x_2|, |y_1 - y_2|\}$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$, is a distance.

(iv) There exists $A \subset \mathbb{R}$ such that both A and its complement $A^\complement$ are dense in $\mathbb{R}$.

(v) $\big(C^1([-1,1]), \|\cdot\|_{C^0}\big)$ is a Banach space, i.e., a complete normed space.

(vi) The complement of a 2^{nd} category set is a 1^{st} category set.

(vii) A nowhere dense set is meagre.

(viii) A meagre set is nowhere dense.

(ix) Let U be the set of fattened rationals in $\mathbb{R}$, namely

$$U := \bigcap_{j=1}^{\infty} U_j, \qquad U_j := \bigcup_{k=1}^{\infty} \Big(q_k - 2^{-(j+k+1)}, q_k + 2^{-(j+k+1)} \Big),$$

where $(q_n)_{n\in\mathbb{N}}$ is a counting of $\mathbb{Q}$. Then $U = \mathbb{Q}$.

2.3. An application of Baire. Let $f \in C^0([0,\infty))$ be a continuous function satisfying

$$\forall t \in [0,\infty): \ \lim_{n\to\infty} f(nt) = 0.$$

Prove that $\lim_{t\to\infty} f(t) = 0$.

Hint. Apply the Baire Lemma as in the proof of the uniform boundedness principle.

2.4. Compactly supported sequences and their ℓ^∞-completion.

Definition. We denote the space of compactly supported sequences by

$$c_c := \{(x_n)_{n\in\mathbb{N}} \in \ell^\infty \mid \exists N \in \mathbb{N} \ \forall n \geq N: \ x_n = 0\}$$

and the space of sequences converging to zero by

$$c_0 := \{(x_n)_{n\in\mathbb{N}} \in \ell^\infty \mid \lim_{n\to\infty} x_n = 0\}.$$

(i) Show that $(c_c, \|\cdot\|_{\ell^\infty})$ is *not* complete. What is the completion of this space?

(ii) Prove the strict inclusion

$$\bigcup_{p\geq 1} \ell^p \subsetneq c_0.$$

2.5. (Dis)-continuity of functions arising as pointwise limits.

Let (X,d) be a metric space, and let (f_n) be a sequence of continuous, real-valued functions $f_n \colon X \to \mathbb{R}$ assumed to be pointwise converging to a limit function f, i.e., we set

$$f(x) = \lim_{n\to\infty} f_n(x).$$

In this problem we wish to study the set of continuity points of the function f, namely the structure of the set $C := \{x \in X \mid f \text{ is continuous at } x\}$.

(i) Give an example of a space X and a sequence of continuous functions whose pointwise limit (although well-defined) is *not* continuous.

(ii) Assuming that (X,d) is complete, prove that C is residual and dense.

Hint. For every $\varepsilon > 0$, define $D_\varepsilon := \{x \in X \mid \operatorname{osc}_x(f) \geq \varepsilon\}$, where $\operatorname{osc}_x(f) := \lim_{r\to 0}\{\sup_{y\in B_r(x)} f(y) - \inf_{y\in B_r(x)} f(y)\}$ is the oscillation of f at x, and set $D := \bigcup_{j\geq 1} D_{1/j}$. Show that: (a) D is the set of discontinuity points of f, i.e. $D^\complement = C$, and (b) $D_\varepsilon^\complement$ is open and dense for all $\varepsilon > 0$. Hence conclude by applying Baire's Lemma.

(iii) Show that the Dirichlet function $f = \chi_{\mathbb{Q}}$ is not the pointwise limit of any sequence of continuous functions on the real line.

2. Solutions

Solution of 2.1: For a metric space (M,d) we shall prove equivalence of (i), (ii), (iii) and (iv).

"(i) ⇒ (ii)" Let $A\subset M$ be a meagre set. Then, $A^{\complement}$ is residual and dense in M by (i). Hence, $\emptyset=(M\setminus A^{\complement})^\circ=A^\circ$.

"(ii) ⇒ (iii)" Let $A\subset M$ be open and meagre. Then $A=A^\circ$ and $A^\circ=\emptyset$ by (ii).

"(iii) ⇒ (iv)" Let $A=\bigcap_{n\in\mathbb{N}}A_n$ be a countable intersection of dense open sets $A_n\subset M$. Since A_n is dense, $(A_n^{\complement})^\circ=\emptyset$. Since A_n is open, $A_n^{\complement}$ is closed. Thus, $(\overline{A_n^{\complement}})^\circ=(A_n^{\complement})^\circ=\emptyset$, which means that $A_n^{\complement}$ is nowhere dense. Thus, $A^{\complement}=\bigcup_{n\in\mathbb{N}}A_n^{\complement}$ is meagre. As a result, $(A^{\complement})^\circ$ is open and meagre, hence empty by (iii). This implies that A is dense in M.

"(iv) ⇒ (i)" Let $\Omega\subset M$ be a residual set. Since $A=\Omega^{\complement}$ is meagre, $A=\bigcup_{n\in\mathbb{N}}A_n$ for nowhere dense sets A_n. Then $\emptyset=(\overline{A_n})^\circ=(M\setminus(\overline{A_n})^{\complement})^\circ$ which implies that $(\overline{A_n})^{\complement}$ is dense in M. Moreover, $(\overline{A_n})^{\complement}$ is open since $\overline{A_n}$ is closed. Then, (iv) implies density of

$$\Omega=A^{\complement}=\bigcap_{n\in\mathbb{N}}A_n^{\complement}\supseteq\bigcap_{n\in\mathbb{N}}(\overline{A_n})^{\complement}.$$

Solution of 2.2:

(i) *False.* Consider $f_n(x)=x^n$. Then $f_n\in C^0([0,1])$ but

$$\lim_{n\to\infty}f_n(x)=f(x):=\begin{cases}0, & \text{if } 0\le x<1,\\ 1, & \text{if } x=1.\end{cases}$$

(ii) *False.* Consider $f_n\colon[0,1]\to\mathbb{R}$ given by (see Figure 1)

$$f_n(x)=\begin{cases}n^2x & \text{if } x\in[0,\frac{1}{n}],\\ 2n-n^2x & \text{if } x\in(\frac{1}{n},\frac{2}{n}],\\ 0 & \text{else.}\end{cases}$$

Then, $f_n(0)=0$ for all $n\in\mathbb{N}$ and $\forall x\in(0,1]\ \forall n\ge\frac{2}{x}:\ f_n(x)=0$. Being convergent to zero, $(f_n(x))_{n\in\mathbb{N}}$ is bounded for all $x\in[0,1]$. However, $\sup\limits_{x\in[0,1]}|f_n(x)|=n$ is unbounded.

(iii) *False.* $d((0,0),(0,y))=\min\{0,|y|\}=0$ for all $y\in\mathbb{R}$.

(iv) *True.* The rationals $\mathbb{Q}\subset\mathbb{R}$ and the irrationals $\mathbb{R}\setminus\mathbb{Q}$ are both dense in $\mathbb{R}$.

(v) *False.* Consider $h_n(x)=\sqrt{x^2+n^{-2}}$. Then $h_n\in C^1([-1,1])$ for every $n\in\mathbb{N}$. $\max_{x\in[-1,1]}|h_n(x)-h_m(x)|=|h_n(0)-h_m(0)|=|\frac{1}{n}-\frac{1}{m}|$ implies that $(h_n)_{n\in\mathbb{N}}$ is Cauchy w.r.t. $\|\cdot\|_{C^0}$ but the C^0-limit function $h(x)=|x|$ is not in $C^1([-1,1])$ (see Figure 1).

(vi) *False.* Both, $(-\infty,0)\subset\mathbb{R}$ and $[0,\infty)\subset\mathbb{R}$ are 2^{nd} category sets.

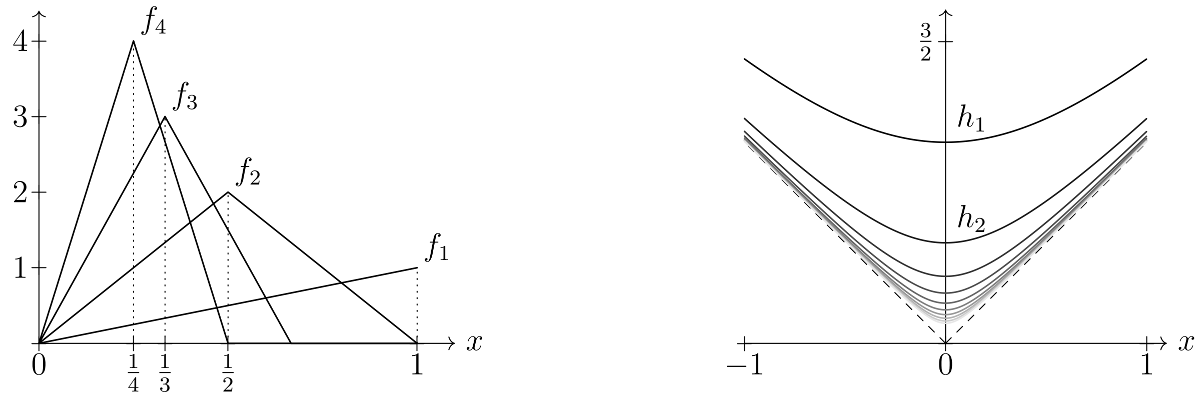

Figure 1: The counterexamples for Problem 2.2, points (ii) and (v).

(vii) *True.* A nowhere dense A can be written as $A = \bigcup_{j=1}^{\infty} A_j$ with $A_j = A$ for all j and thus it is meagre by definition.

(viii) *False.* $\mathbb{Q} = \bigcup_{x\in\mathbb{Q}}\{x\} \subset \mathbb{R}$ is meagre. If $\mathbb{Q}$ were nowhere dense, then the interior of the closure $\overline{\mathbb{Q}}$ would be empty but $\overline{\mathbb{Q}} = \mathbb{R}$.

(ix) *False.* One can use the Baire category to distinguish $\mathbb{Q}$ from U since in fact $\operatorname{Cat}(\mathbb{Q}) = 1$, which follows straight from the definition, and $\operatorname{Cat}(U) = 2$, which we prove in the following. The sets $U_j \subset \mathbb{R}$ are open as unions of open intervals and dense since $\mathbb{Q} \subset U_j$. Therefore, the complements $U_j^{\complement}$ are closed with empty interior, i.e., nowhere dense. Hence, $U^{\complement} = \bigcup_{j=1}^{\infty} U_j^{\complement}$ is of 1^{st} category which implies $\operatorname{Cat}(U) = 2$.

Solution of 2.3: Given $f \in C^0([0,\infty))$ satisfying $\forall t \in [0,\infty) : \lim_{n\to\infty} f(nt) = 0$ we define $f_n(t) = |f(nt)|$ for every $n \in \mathbb{N}$. Let $\varepsilon > 0$ and let

$$A_N := \bigcap_{n=N}^{\infty} \{t \in [0,\infty) \mid f_n(t) \leq \varepsilon\}.$$

Since f_n is continuous, the pre-image $f_n^{-1}([0,\varepsilon]) = \{t \in [0,\infty) \mid f_n(t) \leq \varepsilon\}$ is closed for all $n \in \mathbb{N}$. Thus, the set A_N is closed as intersection of closed sets. By assumption,

$$\forall t \in [0,\infty) \quad \exists N_t \in \mathbb{N} \quad \forall n \geq N_t : \quad f_n(t) \leq \varepsilon,$$

which implies

$$[0,\infty) = \bigcup_{N=1}^{\infty} A_N.$$

The Baire Lemma applied to the complete metric space $([0,\infty), |\cdot|)$ implies that there exists $N_0 \in \mathbb{N}$ such that A_{N_0} has non-empty interior, i.e., there exist $0 \leq a < b$ such that $(a,b) \subset A_{N_0}$. This implies

$$\begin{aligned} &\forall n \geq N_0 \quad \forall t \in (a,b) : \qquad f_n(t) \leq \varepsilon \\ \Leftrightarrow \quad &\forall n \geq N_0 \quad \forall t \in (na,nb) : \ |f(t)| \leq \varepsilon. \end{aligned}$$

If $n > \frac{a}{b-a}$, then $(n+1)a < nb$. For the intervals $J_{a,b}(n) := (na, nb)$ this means that $J_{a,b}(n) \cap J_{a,b}(n+1) \neq \emptyset$. Let $N_1 > \max\{N_0, \frac{a}{b-a}\}$. Then, in particular,

$$\forall t > N_1 a : \qquad |f(t)| \leq \varepsilon.$$

This proves $\lim_{t\to\infty} f(t) = 0$ since $\varepsilon > 0$ was arbitrary.

Solution of 2.4:

(i) Let $x^{(k)} = (x_n^{(k)})_{n\in\mathbb{N}} \in c_c$ be given by

$$x_n^{(k)} = \begin{cases} \frac{1}{n} & \text{for } n \leq k, \\ 0 & \text{for } n > k. \end{cases}$$

Then $(x^{(k)})_{k\in\mathbb{N}}$ is a Cauchy sequence in $(c_c, \|\cdot\|_{\ell^\infty})$. However, its limit sequence $x^{(\infty)}$ given by $x_n^{(\infty)} = \frac{1}{n}$ for all $n \in \mathbb{N}$ is not in c_c but in $c_0 \setminus c_c$. We claim that c_0 is the completion of $(c_c, \|\cdot\|_{\ell^\infty})$.

Proof. It suffices to show $c_0 = \overline{c_c}$, where the closure is taken in ℓ^∞ because then, $(c_0, \|\cdot\|_{\ell^\infty})$ is complete as closed subspace of the complete space $(\ell^\infty, \|\cdot\|_{\ell^\infty})$.

"$\subseteq$" Let $x = (x_n)_{n\in\mathbb{N}} \in c_0$. Let $(x^{(k)})_{k\in\mathbb{N}}$ be a sequence of sequences $x^{(k)} = (x_n^{(k)})_{n\in\mathbb{N}}$ in c_c given by

$$x_n^{(k)} = \begin{cases} x_n & \text{for } n \leq k, \\ 0 & \text{for } n > k. \end{cases}$$

Let $\varepsilon > 0$. By assumption, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_n| < \varepsilon$ for every $n \geq N_\varepsilon$.

$$\Rightarrow\ \forall k \geq N_\varepsilon : \quad \|x^{(k)} - x\|_{\ell^\infty} = \sup_{n>k} |0 - x_n| \leq \varepsilon.$$

We conclude that $x^{(k)} \to x$ in ℓ^∞ as $k \to \infty$ and since $x \in c_0$ is arbitrary, $c_0 \subseteq \overline{c_c}$.

"$\supseteq$" Let $x = (x_n)_{n\in\mathbb{N}} \in \overline{c_c}$. Then there exists a sequence $(x^{(k)})_{k\in\mathbb{N}}$ of sequences $x^{(k)} = (x_n^{(k)})_{n\in\mathbb{N}} \in c_c$ such that $x^{(k)} \to x$ in ℓ^∞ as $k \to \infty$. Let $\varepsilon > 0$. In particular, there exists $K \in \mathbb{N}$ such that

$$\sup_{n\in\mathbb{N}} |x_n^{(K)} - x_n| = \|x^{(K)} - x\|_{\ell^\infty} < \varepsilon$$

Since $x^{(K)} \in c_c$ there exists $N_0 \in \mathbb{N}$ such that $x_n^{(K)} = 0$ for all $n \geq N_0$. This implies that

$$\forall n \geq N_0 : \quad |x_n| \leq \sup_{n\geq N_0} |0 - x_n| < \varepsilon.$$

We conclude that $x_n \to 0$ as $n \to \infty$ which means that $x \in c_0$.

□

(ii) If $(x_n)_{n\in\mathbb{N}} \in \ell^p$ for any $p \geq 1$, then necessarily $x_n \to 0$ for $n \to \infty$ by standard facts concerning summable series. Consequently,

$$\bigcup_{p\geq 1} \ell^p \subset c_0.$$

The inclusion is strict, since $y = (y_n)_{n\in\mathbb{N}} \in c_0$ given by

$$y_n = \frac{1}{\log(n+1)}$$

has the property that $y \notin \ell^p$ for any $p \geq 1$. Indeed, given any $p \geq 1$ there exists $N_p \in \mathbb{N}$ such that $\log(n+1) \leq n^{\frac{1}{p}}$ for every $n \geq N_p$ which allows the estimate

$$\sum_{n=1}^{\infty} \Big(\frac{1}{\log(n+1)}\Big)^p \geq \sum_{n=N_p}^{\infty} \Big(\frac{1}{n^{\frac{1}{p}}}\Big)^p = \sum_{n=N_p}^{\infty} \frac{1}{n} = \infty.$$

Solution of 2.5:

(i) Consider as metric space the interval $[-1,1]$ with the Euclidean metric and define the continuous functions

$$f_n(x) := \begin{cases} 0, & \text{for } x \leq 0 \\ nx, & \text{for } 0 < x < 1/n \\ 1, & \text{for } x \geq 1/n. \end{cases}$$

Then it is very easy to check that the functions f_n pointwise converge to the function

$$f(x) := \begin{cases} 0, & \text{for } x \leq 0 \\ 1, & \text{for } x > 0, \end{cases}$$

which is not continuous at 0.

(ii) First observe that $x \in X$ is a continuity point for f if and only if

$$\operatorname{osc}_x(f) = \lim_{r\to 0} \Big\{ \sup_{y\in B_r(x)} f(y) - \inf_{y\in B_r(x)} f(y) \Big\} = 0.$$

Hence we directly obtain that $D^{\complement} = C$. Moreover $\operatorname{osc}_x(f)$ is upper semicontinuous with respect to the variable x (see Lemma below), which implies that D_ε is closed for all $\varepsilon > 0$ and thus $D_\varepsilon^{\complement}$ is open for all $\varepsilon > 0$.

Lemma. *Let (X,d) be a metric space and let $f\colon X \to \mathbb{R}$ be any function. Then the oscillation $\operatorname{osc}_x(f)$ is upper semicontinuous with respect to the variable x.*

Proof. Let us first prove that, given any function $f\colon X \to \mathbb{R}$, the function $g(x) := \lim_{r\to 0} \sup_{y\in B_r(x)} f(y)$ is upper semicontinuous. Fix any $x \in X$. By definition of g, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $f(y) \leq g(x) + \varepsilon$ for all $y \in B_\delta(x)$. Therefore, for every $y \in B_{\delta/2}(x)$, we obtain that

$$g(y) = \lim_{r\to 0} \sup_{z\in B_r(y)} f(z) \leq \sup_{z\in B_{\delta/2}(y)} f(z) \leq g(x) + \varepsilon,$$

since $B_{\delta/2}(y) \subset B_\delta(x)$. By taking the limit superior as $y \to x$, this implies that $\limsup_{y\to x} g(y) \leq g(x) + \varepsilon$ and, by arbitrariness of ε, this is sufficient to prove the upper semicontinuity of g.

Now observe that $\operatorname{osc}_x(f)$ is the sum of $\lim_{r\to 0} \sup_{y\in B_r(x)} f(y)$ and $-\lim_{r\to 0} \inf_{y\in B_r(x)} f(y) = \lim_{r\to 0} \sup_{y\in B_r(x)}(-f(y))$, which are both upper semicontinuous thanks to the argument above; hence, the oscillation is upper semicontinuous as well. □

We now want to prove that $D_\varepsilon^\complement$ is dense for all fixed $\varepsilon > 0$. For every $k \in \mathbb{N}$, define $E_k := \bigcap_{i,j\geq k}\{x \in X \mid |f_j(x) - f_i(x)| \leq \varepsilon/4\}$. Note that E_k is closed for all $k \in \mathbb{N}$, since the functions f_n are continuous, and that $\bigcup_{k\in\mathbb{N}} E_k = X$, because the functions f_n pointwise converge to f. As a result, by Baire's Lemma, for every open set $U \subset X$ there exists $k \in \mathbb{N}$ with $E_k^\circ \cap U \neq \emptyset$. In particular there exists an open set $V \subset X$ such that $V \subset E_k \cap U$. Hence, by definition of E_k, we have $|f_j(x) - f_i(x)| \leq \varepsilon/4$ for all $x \in V$ and for all $i, j \geq k$. Taking $i = k$ and the limit as $j \to \infty$, this implies that $|f(x) - f_k(x)| \leq \varepsilon/4$ for every $x \in V$. Since f_k is continuous, up to taking V possibly smaller, we can also assume that $|f_k(x) - f_k(y)| \leq \varepsilon/4$ for all $x, y \in V$. Therefore, for all $x, y \in V$, we obtain that

$$|f(x) - f(y)| \leq |f(x) - f_k(x)| + |f_k(x) - f_k(y)| + |f_k(y) - f(y)| \leq \frac{3\varepsilon}{4},$$

which implies that $\operatorname{osc}_x(f) \leq 3\varepsilon/4 < \varepsilon$ for all $x \in X$ and thus $V \subset D_\varepsilon^\complement$. In particular $D_\varepsilon^\complement \cap U \neq \emptyset$ for all open subset $U \subset X$, which means that $D_\varepsilon^\complement$ is dense, as desired.

As a result, we have that $C = \bigcap_{j\geq 1} D_{1/j}^\complement$ is a countable intersection of open dense sets, hence it is residual and dense by Baire's Lemma.

Note. The expression $\lim_{r\to 0} \sup_{y\in B_r(x)} f(y)$ *in the definition of oscillation differs from* $\limsup_{y\to x} f(y)$*, since in the definition of limit superior we do not take into account the value of the function f at the point x; namely* $\limsup_{y\to x} f(y) := \lim_{r\to 0} \sup_{y\in B_r(x)\setminus\{x\}} f(y)$.

(iii) Since both the rational and the irrational numbers are dense in $\mathbb{R}$, we have that f is nowhere continuous. Hence f cannot be the pointwise limit of any sequence of continuous functions on the real line, because otherwise the set of continuity points of f would be dense by (ii).