3.1. Bounded maps to Banach space ✍. Let M be a set and let $(X, \|\cdot\|_X)$ be a Banach space. Then show that the set of bounded maps

$$B(M,X) := \left\{ f \colon M \to X \mid \sup_{m\in M} \|f(m)\|_X < \infty \right\}$$

endowed with the norm

$$|f| = \sup_{m\in M} \|f(m)\|_X$$

(as defined in class) is itself a Banach space.

3.2. Normal convergence ✍. Let $(X, \|\cdot\|)$ be a normed vector space. Prove that the following statements are equivalent.

(i) $(X, \|\cdot\|)$ is a Banach space.

(ii) For every sequence $(x_n)_{n\in\mathbb{N}}$ in X with $\sum_{k=1}^{\infty} \|x_n\| < \infty$, the limit $\lim_{N\to\infty} \sum_{n=1}^{N} x_n$ exists.

Hint. A Cauchy sequence converges if and only if it has a convergent subsequence.

3.3. Subsets with compact boundary ⚙. Let $(X, \|\cdot\|)$ be an infinite-dimensional normed vector space and let $Z \subset X$ be a bounded subset with compact boundary. Prove that Z has empty interior, i.e., $Z^\circ = \emptyset$.

Hint. Assume that $Z^\circ \neq \emptyset$. Find a continuous map that projects the boundary ∂Z to the boundary of a ball inside Z. This will contradict the fact that the unit sphere in an infinite-dimensional normed space is non-compact.

3.4. Topological complement ⚙.

Definition. Let $(X, \|\cdot\|_X)$ be a Banach space. A subspace $U \subset X$ is called *topologically complemented* if there is a subspace $V \subset X$ such that the linear map I given by

$$\begin{aligned} I \colon (U \times V, \|\cdot\|_{U\times V}) &\to (X, \|\cdot\|_X), \\ (u,v) &\mapsto u+v \end{aligned} \qquad \|(u,v)\|_{U\times V} := \|u\|_X + \|v\|_X,$$

is a continuous isomorphism of normed spaces with continuous inverse. In this case V is said to be a *topological complement* of U.

(i) Prove that $U \subset X$ is topologically complemented if and only if there exists a continuous linear map $P \colon X \to X$ with $P \circ P = P$ and image $P(X) = U$.

(ii) Show that a topologically complemented subspace must be closed.

3.5. Closed subspaces ⚙ 💎. Show that the subspaces

$$U = \{(x_n)_{n\in\mathbb{N}} \in \ell^1 \mid \forall n \in \mathbb{N}: \; x_{2n} = 0\},$$
$$V = \{(x_n)_{n\in\mathbb{N}} \in \ell^1 \mid \forall n \in \mathbb{N}: \; x_{2n-1} = n x_{2n}\}$$

are both closed in $(\ell^1, \|\cdot\|_{\ell^1})$ while the subspace $U \oplus V$ is not closed in $(\ell^1, \|\cdot\|_{\ell^1})$.

Hint. Prove that if any sequence $(x^{(k)})_{k\in\mathbb{N}}$ of elements $x^{(k)} = (x_n^{(k)})_{n\in\mathbb{N}} \in \ell^1$ converges to $(x_n)_{n\in\mathbb{N}}$ in ℓ^1 for $k \to \infty$, then each entry $x_n^{(k)}$ converges in $\mathbb{R}$ to x_n for $k \to \infty$. For the second claim, show $c_c \subset U \oplus V$. (Recall c_c from Problem 2.4.)

3. Solutions

Solution of 3.1: We need to check that $B(M,X)$ is complete with respect to the topology induced by the norm $|\cdot|$. Let $(f_k)_{k\in\mathbb{N}}$ be a Cauchy sequence, i.e., for every $\varepsilon>0$ there exists $K\in\mathbb{N}$ such that $|f_k-f_h|\leq\varepsilon$ for all $k,h\geq K$. Then note that the sequence $(f_k(m))_{k\in\mathbb{N}}$ is Cauchy in $(X,\|\cdot\|_X)$ for all $m\in M$, since $|f_k(m)-f_h(m)|\leq\sup_{n\in M}\|f_k(n)-f_h(n)\|_X=|f_k-f_h|$. Hence we can define a function $f\colon M\to X$ as the pointwise limit of the sequence $(f_k)_{k\in\mathbb{N}}$, namely $f(m):=\lim_{k\to\infty}f_k(m)$, which exists for all $m\in M$ because $(f_k(m))_{k\in\mathbb{N}}$ is Cauchy in the Banach space $(X,\|\cdot\|_X)$.

Finally we need to show that $f\in B(M,X)$ and that the sequence $(f_k)_{k\in\mathbb{N}}$ converges to f with respect to the norm $|\cdot|$. To this purpose, fixed any $\varepsilon>0$, observe that there exists $K\in\mathbb{N}$ with

$$|f_k-f_h|\leq\varepsilon\ \forall k,h\geq K \implies \|f_k(m)-f_h(m)\|_X\leq\varepsilon\ \forall k,h\geq K\ \forall m\in M.$$

Letting $h\to\infty$, this implies

$$\|f(m)-f_k(m)\|_X\leq\varepsilon\ \forall k\geq K\ \forall m\in M \implies |f-f_k|\leq\varepsilon\ \forall k\geq K,$$

which proves that $(f_k)_{k\in\mathbb{N}}$ converges to $f\in B(M,X)$.

Solution of 3.2: If $(X,\|\cdot\|)$ is a Banach space, and $(x_k)_{k\in\mathbb{N}}$ any sequence in X with $\sum_{k=1}^{\infty}\|x_k\|<\infty$, then $(s_n)_{n\in\mathbb{N}}$ given by $s_n=\sum_{k=1}^{n}x_k$ is a Cauchy sequence (and hence convergent) since by assumption, every $\varepsilon>0$ allows $N_\varepsilon\in\mathbb{N}$ such that for every $m\geq n\geq N_\varepsilon$,

$$\|s_m-s_n\|\leq\sum_{k=n+1}^{m}\|x_k\|\leq\sum_{k=N_\varepsilon+1}^{\infty}\|x_k\|<\varepsilon.$$

Conversely, we assume for every sequence $(x_k)_{k\in\mathbb{N}}$ in X that $\sum_{k=1}^{\infty}\|x_k\|<\infty$ implies convergence of $s_n=\sum_{k=1}^{n}x_k$ in X for $n\to\infty$. Let $(y_n)_{n\in\mathbb{N}}$ be a Cauchy in X. Then,

$$\forall k\in\mathbb{N}\quad\exists N_k\in\mathbb{N}\quad\forall n,m\geq N_k:\quad\|y_n-y_m\|\leq 2^{-k}.$$

Without loss of generality, we can assume $N_{k+1}>N_k$. Let $x_k:=y_{N_{k+1}}-y_{N_k}$. Then,

$$\sum_{k=1}^{\infty}\|x_k\|=\sum_{k=1}^{\infty}\|y_{N_{k+1}}-y_{N_k}\|\leq\sum_{k=1}^{\infty}2^{-k}<\infty,$$

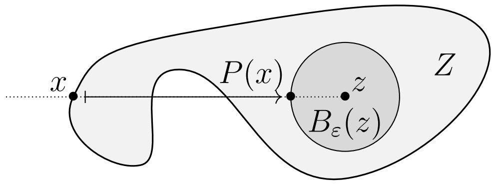
which by assumption implies that

$$s_n=\sum_{k=1}^{n}x_k=\sum_{k=1}^{n}(y_{N_{k+1}}-y_{N_k})=y_{N_{n+1}}-y_{N_1}$$

converges in X for $n \to \infty$. Hence, $(y_{N_n})_{n\in\mathbb{N}}$ is a convergent subsequence of $(y_n)_{n\in\mathbb{N}}$. Since $(y_n)_{n\in\mathbb{N}}$ is Cauchy, it converges to the same limit in X. Thus, X is complete.

Solution of 3.3: If $Z \subset X$ has non-empty interior $Z^\circ \neq \emptyset$, then there exists $z \in Z$ and $\varepsilon > 0$ such that $B_\varepsilon(z) \subset Z^\circ$, where $B_\varepsilon(z)$ denotes the ball of radius ε around z in $(X, \|\cdot\|)$ and $\partial B_\varepsilon(z)$ its boundary. We consider the projection

$$\begin{aligned} P\colon Z \setminus \{z\} &\to \partial B_\varepsilon(z) \\ x &\mapsto z + \varepsilon \frac{x - z}{\|x - z\|}. \end{aligned}$$



For every $y \in \partial B_\varepsilon$ the ray $\gamma = \{z + t(y - z) \mid t > 0\}$ must intersect ∂Z since Z is assumed to be bounded. Therefore, $P(\partial Z) = \partial B_\varepsilon(z)$. Being continuous, P maps compact sets onto compact sets. Since ∂Z is assumed to be compact, we have that the sphere $\partial B_\varepsilon(z)$ is compact. This however contradicts the assumption that the dimension of X is infinite.

Solution of 3.4:

(i) Suppose, $U \subset X$ is topologically complemented by $V \subset X$. Then, $I\colon U \times V \to X$ with $(u, v) \mapsto u + v$ is a continuous isomorphism with continuous inverse. We define

$$\begin{aligned} P_1\colon U \times V &\to U \times V, & P &:= I \circ P_1 \circ I^{-1}\colon X \to X. \\ (u, v) &\mapsto (u, 0) \end{aligned}$$

The map P_1 is linear, bounded since $\|P_1(u, v)\|_{U\times V} = \|u\|_U \leq \|(u, v)\|_{U\times V}$ and hence continuous. As composition of linear continuous maps, P is linear and continuous. Moreover,

$$\begin{aligned} &P \circ P = (I \circ P_1 \circ I^{-1}) \circ (I \circ P_1 \circ I^{-1}) = I \circ P_1 \circ P_1 \circ I^{-1} = I \circ P_1 \circ I^{-1} = P, \\ &P(X) = I(U \times \{0\}) = U. \end{aligned}$$

Conversely, suppose $U \subset X$ allows a continuous linear map $P\colon X \to X$ with $P \circ P = P$ and $P(X) = U$. Let $V := \ker(P)$. Then

$$P \circ (1 - P) = P - P = 0 \qquad \Rightarrow (1 - P)(X) \subseteq \ker(P) = V. \tag{1}$$

In fact, $(1 - P)(X) = V$ since given $v \in V$ we have $v = (1 - P)v$. Analogously,

$$(1 - P) \circ P = P - P = 0 \qquad \Rightarrow\; U = P(X) \subseteq \ker(1 - P). \tag{2}$$

In fact, $U = \ker(1 - P)$ since $x - Px = 0$ implies $x = Px \in U$. The claim is, that

$$\begin{aligned} I\colon U \times V &\to X \\ (u, v) &\mapsto u + v \end{aligned}$$

is continuous and has a continuous inverse. The continuity of I follows directly from

$$\|I(u, v)\|_X = \|u + v\|_X \leq \|u\|_X + \|v\|_X = \|(u, v)\|_{U\times V}.$$

By the assumptions on P, in particular by (1), the map

$$\begin{aligned}\Phi\colon X &\to U\times V\\ x&\mapsto \big(Px,(1-P)x\big)\end{aligned}$$

is well-defined and continuous. Since $Pu=u$ for all $u\in U$ by (2), we have

$$\begin{aligned}(\Phi\circ I)(u,v)&=\Phi(u+v)=\Big(Pu+Pv,u-Pu+v-Pv\Big)=(u,v).\\ (I\circ\Phi)(x)&=I(Px,(1-P)x)=Px+(1-P)x=x,\end{aligned}$$

which implies that Φ is inverse to I. Consequently, U is topologically complemented.

(ii) If $U\subset X$ is topologically complemented, then (i) implies existence of a continuous map $P\colon X\to X$ with $\ker(1-P)=U$. Thus, U must be closed as the kernel of the continuous map $1-P$.

Remark. If X is *not* isomorphic to a Hilbert space, then X has closed subspaces which are *not* topologically complemented (see [Lindenstrauss & Tzafriri. *On the complemented subspaces problem.* (1971)]). An example is $c_0\subset\ell^\infty$ but this is not easy to prove.

Solution of 3.5:

Claim 1. Let $(x^{(k)})_{k\in\mathbb{N}}$ be a sequence of sequences $x^{(k)}=(x_n^{(k)})_{n\in\mathbb{N}}\in\ell^1$ and let $x=(x_n)_{n\in\mathbb{N}}\in\ell^1$. Then, the following implication is true.

$$\lim_{k\to\infty}\|x^{(k)}-x\|_{\ell^1}=0\quad\Rightarrow\ \forall n\in\mathbb{N}:\ \lim_{k\to\infty}|x_n^{(k)}-x_n|=0.$$

Proof. Let $\varepsilon>0$ and $n\in\mathbb{N}$. By assumption, there exists $K_\varepsilon\in\mathbb{N}$ such that

$$\forall k\geq K_\varepsilon:\quad |x_n^{(k)}-x_n|\leq\sum_{n=1}^{\infty}|x_n^{(k)}-x_n|=\|x^{(k)}-x\|_{\ell^1}<\varepsilon.$$ □

Claim 2. $U=\{(x_n)_{n\in\mathbb{N}}\in\ell^1\mid\forall n\in\mathbb{N}:\ x_{2n}=0\}$ is closed in $(\ell^1,\|\cdot\|_{\ell^1})$.

Proof. Let $(x^{(k)})_{k\in\mathbb{N}}$ be a sequence of sequences $x^{(k)}=(x_n^{(k)})_{n\in\mathbb{N}}\in U$ converging to $x=(x_n)_{n\in\mathbb{N}}$ in ℓ^1. By definition, $x_{2n}^{(k)}=0$ for every $n\in\mathbb{N}$. According to Claim 1,

$$x_{2n}=\lim_{k\to\infty}x_{2n}^{(k)}=0$$

for every $n\in\mathbb{N}$. Thus, $x\in U$. □

Claim 3. $V=\{(x_n)_{n\in\mathbb{N}}\in\ell^1\mid\forall n\in\mathbb{N}:\ x_{2n-1}=nx_{2n}\}$ is closed in $(\ell^1,\|\cdot\|_{\ell^1})$.

Proof. Let $(x^{(k)})_{k\in\mathbb{N}}$ be a sequence of sequences $x^{(k)} = (x_n^{(k)})_{n\in\mathbb{N}} \in V$ converging to $x = (x_n)_{n\in\mathbb{N}}$ in ℓ^1. By definition, $x_{2n-1}^{(k)} = nx_{2n}^{(k)}$ for every $n \in \mathbb{N}$. By Claim 1,

$$x_{2n-1} = \lim_{k\to\infty} x_{2n-1}^{(k)} = \lim_{k\to\infty} nx_{2n}^{(k)} = nx_{2n}$$

for every $n \in \mathbb{N}$. Thus, $x \in V$. □

Claim 4. $c_c := \{(x_n)_{n\in\mathbb{N}} \in \ell^\infty \mid \exists N \in \mathbb{N}\ \forall n \geq N:\ x_n = 0\} \subset U \oplus V$.

Proof. Let $x \in c_c$. Then, $x = u + v$ with $u = (u_m)_{m\in\mathbb{N}}$ and $v = (v_m)_{m\in\mathbb{N}}$ given by

$$u_m = \begin{cases} x_m - nx_{m+1}, & \text{if } m = 2n-1, \\ 0, & \text{if } m \text{ is even} \end{cases} \qquad v_m = \begin{cases} nx_{m+1}, & \text{if } m = 2n-1, \\ x_m, & \text{if } m \text{ is even.} \end{cases}$$

The assumption $x \in c_c$ implies $u, v \in c_c \subset \ell^1$. Then, $u \in U$ holds by construction and $v \in V$ follows from $v_{2n-1} = nx_{2n-1+1} = nx_{2n} = nv_{2n}$ for every $n \in \mathbb{N}$. □

Claim 5. The space c_c is dense in $(\ell^1, \|\cdot\|_{\ell^1})$.

Proof. Let $x \in \ell^1$. Let $x^{(k)} = (x_n^{(k)})_{n\in\mathbb{N}} \in c_c$ be given by

$$x_n^{(k)} = \begin{cases} x_n & \text{for } n < k, \\ 0 & \text{for } n \geq k. \end{cases}$$

Then,

$$\|x^{(k)} - x\|_{\ell^1} = \sum_{n=k}^{\infty} |x_n| \xrightarrow{k\to\infty} 0.$$ □

Claim 6. The sequence $x = (x_m)_{m\in\mathbb{N}}$ defined as follows is in ℓ^1 but not in $U \oplus V$.

$$x_m = \begin{cases} 0, & \text{if } m \text{ is odd}, \\ \frac{1}{n^2}, & \text{if } m = 2n. \end{cases}$$

Proof. Since $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$ we have $x \in \ell^1$. Suppose $x = u + v$ for $u \in U$ and $v \in V$. Then, $u_{2n} = 0$ implies $v_{2n} = x_{2n} = \frac{1}{n^2}$ for every $n \in \mathbb{N}$. By definition of V, we have $v_{2n-1} = nv_{2n} = \frac{1}{n}$ for every $n \in \mathbb{N}$. However, $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ implies $v \notin \ell^1$ which contradicts the definition of V. □

Claims 4, 5 and 6 imply that

$$\overline{U \oplus V} \supset \overline{c_c} = \ell^1 \supsetneq U \oplus V.$$

Therefore, $U \oplus V$ cannot be closed.