4.1. Operator norm ✍.

(i) Let $A \in L(\mathbb{R}^n, \mathbb{R}^n)$ be a linear map from $\mathbb{R}^n$ to itself. Show that the squared operator norm $\|A\|^2$ equals the largest eigenvalue of $A^\intercal A$.

(ii) Let $A \in L(\mathbb{R}^{2020}, \mathbb{R}^{2020})$ be symmetric such that there exists a basis $\mathcal{B}$ of $\mathbb{R}^{2020}$ diagonalising A with eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_{2020}\} = \{1, 2, \ldots, 2020\}$ each with multiplicity one.

Let $B \in L(\mathbb{R}^{2020}, \mathbb{R}^{2020})$ be symmetric such that there exists a basis $\mathcal{B}'$ not necessarily equal to $\mathcal{B}$ of $\mathbb{R}^{2020}$ diagonalising B with eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_{2020}\} = \{1, 2, \ldots, 2020\}$ each with multiplicity one.

Prove that the operator norm of the composition BA can be estimated by

$$\|BA\| < 4\,410\,000.$$

4.2. Right shift operator ✍. The right shift map on the space ℓ^2 is given by

$$\begin{aligned} S\colon \ell^2 &\to \ell^2 \\ (x_1, x_2, \ldots) &\mapsto (0, x_1, x_2, \ldots). \end{aligned}$$

(i) Show that that the map S is a continuous linear operator with norm $\|S\| = 1$.

(ii) Compute the eigenvalues and the spectral radius of S.

(iii) Show that S has a left inverse in the sense that there exists an operator $T\colon \ell^2 \to \ell^2$ with $T \circ S = \mathrm{id}\colon \ell^2 \to \ell^2$. Check that $S \circ T \neq \mathrm{id}$.

4.3. Volterra equation ✍. Let $k\colon [0,1] \times [0,1] \to \mathbb{R}$ be continuous. Show that for every $g \in C^0([0,1])$ there exists a unique $f \in C^0([0,1])$ satisfying

$$\forall t \in [0,1]: \quad f(t) + \int_0^t k(t,s) f(s) \,\mathrm{d}s = g(t).$$

Hint. Choose a space $(X, \|\cdot\|_X)$ and show that the operator $T\colon X \to X$ given by

$$(Tf)(t) = \int_0^t k(t,s) f(s) \,\mathrm{d}s$$

has spectral radius $r_T = 0$. Then apply Satz 2.2.7 (Struwe's notes).

4.4. Unbounded map and approximations ✍. As in Problem 2.4, we denote the space of compactly supported sequences by

$$c_c := \{(x_n)_{n\in\mathbb{N}} \in \ell^\infty \mid \exists N \in \mathbb{N}\ \forall n \geq N: \ x_n = 0\}$$

endowed with the norm $\|\cdot\|_{\ell^\infty}$. Consider the map

$$\begin{aligned} T\colon c_c &\to c_c \\ (x_n)_{n\in\mathbb{N}} &\mapsto (n x_n)_{n\in\mathbb{N}} \end{aligned}$$

(i) Show that T is not continuous.

(ii) Construct continuous linear maps $T_m\colon c_c \to c_c$ such that

$$\forall x \in c_c : \quad T_m x \xrightarrow{m\to\infty} Tx.$$

4.5. Continuity of bilinear maps. Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be normed spaces. We consider the space $(X \times Y, \|\cdot\|_{X\times Y})$, where $\|(x, y)\|_{X\times Y} = \|x\|_X + \|y\|_Y$ and a bilinear map $B\colon X \times Y \to Z$.

(i) Show that B is continuous if

$$\exists C > 0 \quad \forall (x, y) \in X \times Y : \quad \|B(x, y)\|_Z \le C\|x\|_X\|y\|_Y. \tag{†}$$

(ii) Assume that $(X, \|\cdot\|_X)$ is complete. Assume further that the maps

$$\begin{aligned} X &\to Z \\ x &\mapsto B(x, y') \end{aligned} \qquad\qquad \begin{aligned} Y &\to Z \\ y &\mapsto B(x', y) \end{aligned}$$

are continuous for every $x' \in X$ and $y' \in Y$. Prove that then, (†) holds.

Hint. Apply the Theorem of Banach-Steinhaus to a suitable map but recall that it requires completeness of the domain.

4. Solutions

Solution of 4.1:

(i) Let $\langle\cdot,\cdot\rangle$ be the the Euclidean scalar product on $\mathbb{R}^n$ and $|\cdot|$ the Euclidean norm. We choose the standard basis and represent $A \in L(\mathbb{R}^n, \mathbb{R}^n)$ by a matrix which we denote also by A. Let $A^\intercal$ be the transposed matrix. From the definition follows that

$$\begin{aligned}\|A\|^2 &= \sup\{|Ax|^2 \mid x \in \mathbb{R}^n,\ |x|^2 = 1\},\\ |Ax|^2 &= \langle Ax, Ax\rangle = (Ax)^\intercal(Ax) = x^\intercal A^\intercal Ax = \langle x, A^\intercal Ax\rangle.\end{aligned}$$

Recall that $A^\intercal A$ is a symmetric matrix and therefore diagonalizable by an orthonormal basis of eigenvectors $e_1, \ldots, e_n$ with real eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$. Let $x \in \mathbb{R}^n$ with $|x|^2 = 1$ be given. Then there exist $x_1, \ldots, x_n \in \mathbb{R}$ such that $x = x_1e_1 + \ldots + x_ne_n$ and $x_1^2 + \ldots + x_n^2 = 1$. From

$$\langle x, A^\intercal Ax\rangle = \Big\langle x, A^\intercal A\sum_{i=1}^n x_ie_i\Big\rangle = \Big\langle x, \sum_{i=1}^n x_i\lambda_ie_i\Big\rangle = \sum_{i=1}^n \lambda_ix_i^2 \leq \lambda_n\sum_{i=1}^n x_i^2 = \lambda_n$$

we conclude $\|A\|^2 \leq \lambda_n$. Since $\langle e_n, A^\intercal Ae_n\rangle = \langle e_n, \lambda_ne_n\rangle = \lambda_n$, we have $\|A\|^2 = \lambda_n$.

(ii) Since A and B are assumed to be symmetric, we have $A^\intercal A = A^2$ and $B^\intercal B = B^2$. In the basis $\mathcal{B}$ respectively $\mathcal{B}'$ we see that $(2020)^2$ is the largest eigenvalue of A^2 respectively B^2. Using (i), we have $\|A\| = 2020 = \|B\|$. Since $|By| \leq \|B\||y|$ for all $y \in \mathbb{R}^n$ and in particular for $y = Ax$, we have

$$\|BA\| = \sup_{|x|=1}|BAx| \leq \sup_{|x|=1}\|B\||Ax| = \|B\|\sup_{|x|=1}|Ax| = \|B\|\|A\| \leq (2020)^2.$$

To conclude, we notice that $(2020)^2 < (2100)^2 = 21^2 \cdot 10^4 = 441 \cdot 10^4$.

Solution of 4.2:

(i) Let $x \in \ell^2$. By definition of S and ℓ^2-norm, we have $\|Sx\|_{\ell^2} = \|x\|_{\ell^2}$, which implies $\|S\| = 1$. Being linear and bounded, the map S is continuous.

(ii) Suppose $x = (x_n)_{n\in\mathbb{N}} \in \ell^2$ satisfies $Sx = \lambda x$ for some $\lambda \in \mathbb{R}$. Then

$$(0, x_1, x_2, \ldots) = (\lambda x_1, \lambda x_2, \lambda x_3 \ldots).$$

If $\lambda = 0$, then $x = 0$ is immediate. If $\lambda \neq 0$, then $x = 0$ follows via

$$0 = \lambda x_1 \ \Rightarrow\ 0 = x_1 = \lambda x_2 \ \Rightarrow\ 0 = x_2 = \lambda x_3 \ \Rightarrow\ \ldots$$

We conclude that S does not have eigenvalues. The spectral radius of S is

$$r_S = \lim_{n\to\infty}\|S^n\|^{\frac{1}{n}} = 1$$

since $\|S^n\| = 1$ follows for every $n \in \mathbb{N}$ from $\|S^nx\|_{\ell^2} = \|x\|_{\ell^2}$ as in (i).

(iii) We define $T\colon \ell^2 \to \ell^2$ to be the left shift map $T\colon (x_1, x_2, \dots) \mapsto (x_2, x_3, \dots)$. Then, $T \circ S = \mathrm{id}$ and $S \circ T \neq \mathrm{id}$. Indeed,

$$\begin{aligned}(T \circ S)(x_1, x_2, \dots) &= T(0, x_1, x_2, \dots) = (x_1, x_2, \dots),\\ (S \circ T)(x_1, x_2, \dots) &= S(x_2, x_3, \dots) = (0, x_2, x_3, \dots).\end{aligned}$$

Solution of 4.3: Let $(X, \|\cdot\|_X) = \big(C^0([0,1]), \|\cdot\|_{C^0([0,1])}\big)$. Since the function k is continuous in both variables, the integral operator $T\colon X \to X$ given by

$$(Tf)(t) = \int_0^t k(t,s) f(s)\,\mathrm{d}s$$

is well-defined. We claim that, for every $n \in \mathbb{N}$ and every $f \in X$ and $t \in [0,1]$, it holds

$$|(T^n f)(t)| \leq \frac{t^n}{n!} \|k\|^n_{C^0([0,1]\times[0,1])} \|f\|_X.$$

We prove the claim by induction. For $n = 1$ we have

$$|(Tf)(t)| \leq \int_0^t |k(t,s)||f(s)|\,\mathrm{d}s \leq t\|k\|_{C^0([0,1]\times[0,1])}\|f\|_X.$$

Suppose the claim is true for some $n \in \mathbb{N}$. Then,

$$\begin{aligned}|(T^{n+1} f)(t)| &\leq \int_0^t |k(t,s)||(T^n f)(s)|\,\mathrm{d}s\\ &\leq \frac{1}{n!}\|k\|^{n+1}_{C^0}\|f\|_X \int_0^t s^n\,\mathrm{d}s = \frac{t^{n+1}}{(n+1)!}\|k\|^{n+1}_{C^0}\|f\|_X\end{aligned}$$

which proves the claim. Since $0 \leq t \leq 1$, the claim implies

$$r_T := \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} \leq \lim_{n\to\infty} \frac{\|k\|_{C^0}}{(n!)^{\frac{1}{n}}} = 0.$$

From $r_T = 0$ we conclude that the operator $(1 + T) = (1 - (-T))$ is invertible with bounded inverse (Satz 2.2.7 in Struwe's notes). The solution to the Volterra equation $f + Tf = g$ is then given by $f = (1 + T)^{-1} g$.

Solution of 4.4:

(i) The operation $T\colon (x_n)_{n\in\mathbb{N}} \mapsto (n x_n)_{n\in\mathbb{N}}$ is linear in each entry and therefore linear as map $T\colon c_c \to c_c$. For every $k \in \mathbb{N}$ we define the sequence $e^{(k)} = (e^{(k)}_n)_{n\in\mathbb{N}} \in c_c$ by

$$e^{(k)}_n = \begin{cases} 1, & \text{if } n = k,\\ 0, & \text{otherwise.}\end{cases}$$

Then, $\|e^{(k)}\|_{\ell^\infty} = 1$ for every $k \in \mathbb{N}$ but $\|Te^{(k)}\|_{\ell^\infty} = k$ is unbounded for $k \in \mathbb{N}$. As unbounded linear map, T is not continuous.

(ii) For every $m \in \mathbb{N}$ we define

$$\begin{aligned} T_m \colon c_c &\to c_c \\ (x_n)_{n\in\mathbb{N}} &\mapsto (x_1, 2x_2, 3x_3, \ldots, mx_m, 0, 0, \ldots) \end{aligned}$$

Then T_m is linear. $T_m \colon (c_c, \|\cdot\|_\ell^\infty) \to (c_c, \|\cdot\|_\ell^\infty)$ is also bounded for every (fixed) $m \in \mathbb{N}$ since for every $x = (x_n)_{n\in\mathbb{N}} \in c_c$

$$\|T_m x\| = \sup_{n\in\mathbb{N}} |(T_m x)_n| = \max_{n\in\{1,\ldots,m\}} |nx_n| \leq m\|x\|_{\ell^\infty}.$$

Hence, T_m is continuous.

Let $x = (x_n)_{n\in\mathbb{N}} \in c_c$ be fixed. Then there exists $N \in \mathbb{N}$ such that $x_n = 0$ for all $n \geq N$ which implies $T_m x = Tx$ for all $m \geq N$. In particular,

$$T_m x \xrightarrow{m\to\infty} Tx.$$

Solution of 4.5:

(i) Let $((x_k, y_k))_{k\in\mathbb{N}}$ be a sequence in $X \times Y$ converging to (x, y) in $(X \times Y, \|\cdot\|_{X\times Y})$. By definition,

$$\|x_k - x\|_X + \|y_k - y\|_Y = \|(x_k - x, y_k - y)\|_{X\times Y} = \|(x_k, y_k) - (x, y)\|_{X\times Y}$$

which yields convergence $x_k \to x$ in X and $y_k \to y$ in Y. Since $B \colon X \times Y \to Z$ is bilinear, we have

$$\begin{aligned} \|B(x_k, y_k) - B(x, y)\|_Z &= \|B(x_k, y_k) - B(x, y_k) + B(x, y_k) - B(x, y)\|_Z \\ &= \|B(x_k - x, y_k) - B(x, y_k - y)\|_Z \\ &\leq \|B(x_k - x, y_k)\|_Z + \|B(x, y_k - y)\|_Z. \end{aligned}$$

Using the assumption $\|B(x, y)\|_Z \leq C\|x\|_X\|y\|_Y$ and the fact, that convergence of $(y_k)_{k\in\mathbb{N}}$ in $(Y, \|\cdot\|_Y)$ implies that $\|y_k\|_Y$ is bounded uniformly for all $k \in \mathbb{N}$, we conclude

$$\|B(x_k, y_k) - B(x, y)\|_Z \leq C\|x - x_k\|_X\|y_k\|_Y + C\|x\|_X\|y - y_k\|_Y \xrightarrow{k\to\infty} 0.$$

(ii) Let $B_1^Y \subset Y$ be the unit ball around the origin in $(Y, \|\cdot\|_Y)$. For every $x \in X$ we have by assumption

$$\sup_{y'\in B_1^Y} \|B(x, y')\|_Z \leq \sup_{y'\in B_1^Y} \|y'\|_Y \|B(x, \cdot)\|_{L(Y,Z)} \leq \|B(x, \cdot)\|_{L(Y,Z)} < \infty,$$

which means that the maps $(B(\cdot, y'))_{y'\in B_1^Y} \in L(X, Z)$ are pointwise bounded. Since X is assumed to be complete, the Theorem of Banach-Steinhaus implies that $(B(\cdot, y'))_{y'\in B_1^Y} \in L(X, Z)$ are uniformly bounded, i.e.,

$$C := \sup_{y'\in B_1^Y} \|B(\cdot, y')\|_{L(X,Z)} < \infty.$$

From that we conclude

$$\begin{aligned} \|B(x, y)\|_Z &= \|y\|_Y \Big\| B\Big(x, \frac{y}{\|y\|_Y}\Big)\Big\|_Z \\ &\leq \|y\|_Y \|x\|_X \Big\| B\Big(\cdot, \frac{y}{\|y\|_Y}\Big)\Big\|_{L(X,Z)} \leq C\|x\|_X\|y\|_Y. \end{aligned}$$