5.1. Quotient of a Hilbert space. Let $(X, \langle\cdot,\cdot\rangle)$ be a Hilbert space over $\mathbb{C}$ and let $Y \subset X$ be a closed subspace. Prove that the quotient X/Y is isometric to the orthogonal $Y^\perp$ of Y.

5.2. Isomorphic proper subspaces. Let $(X, \langle\cdot,\cdot\rangle)$ be a Hilbert space over $\mathbb{C}$ endowed with a countable Hilbertian basis $\{e_i\}_{i\geq 1}$.

(i) For all $k \geq 1$, consider the subspace $E_k \subset X$ generated by $e_1, e_3, \ldots, e_{2k-1}$, namely $E_k := \langle e_1, e_3, \ldots, e_{2k-1}\rangle_{\mathbb{C}}$ and define $Y := \cup_{k\geq 1} E_k$. Is Y closed?

(ii) Construct a proper subspace $Z \subsetneq X$ such that there exists an isomorphism $T\colon Z \to X$ of Banach spaces.

5.3. Odd and even functions. Consider the Hilbert space $H = L^2((-1,1);\mathbb{R})$ and define the subset of odd functions $D := \{f \in H \mid f(-x) = -f(x) \text{ for a.e. } x \in (-1,1)\}$ and the subset of even functions $P := \{f \in H \mid f(-x) = f(x) \text{ for a.e. } x \in (-1,1)\}$.

(i) Prove that D and P are closed subspaces of H.

(ii) Prove that $H = D \oplus P$ and $D \perp P$. Hence deduce that $D^\perp = P$ and $P^\perp = D$.

(iii) Compute the orthogonal projections $\pi_D\colon H \to D$ and $\pi_P\colon H \to P$.

(iv) Find a Hilbertian basis for both D and P.

5.4. Notable series. Specifying Parseval's identity to $f(x) = x$ (seen as an element of $L^2((-\pi,\pi);\mathbb{R})$) show that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

5.5. Closed sum of subspaces. Let $(X, \|\cdot\|_X)$ be a normed space an let $U, V \subset X$ be subspaces. Prove the following.

(i) If U is finite dimensional and V closed, then $U + V$ is a closed subspace of X.

(ii) If V is closed with finite codimension, i.e., $\dim(X/V) < \infty$, then $U + V$ is closed.

Hint. Is the canonical quotient map $\pi\colon X \to X/V$ continuous? What is $\pi^{-1}(\pi(U))$?

5.6. Vanishing boundary values. Let $X = C^0([0,1])$ and $U = C^0_0([0,1]) := \{f \in C^0([0,1]) \mid f(0) = 0 = f(1)\}$.

(i) Show that U is a closed subspace of X endowed with the norm $\|\cdot\|_X = \|\cdot\|_{C^0([0,1])}$.

(ii) Compute the dimension of the quotient space X/U and find a basis for X/U.

5. Solutions

Solution of 5.1: Consider the orthogonal projection $\pi_{Y^\perp}\colon X \to Y^\perp$ of X onto the closed subspace $Y^\perp$. Note that $x \in \ker \pi_{Y^\perp}$ if and only if $\langle x, z\rangle = 0$ for all $z \in Y^\perp$, if and only if $x \in Y$. Moreover $\pi_{Y^\perp}$ is obviously surjective since $\pi_{Y^\perp}(z) = z$ for all $z \in Y^\perp$. Hence we can consider the map $F\colon X/Y \to Y^\perp$ such that $F([x]) = \pi_{Y^\perp}(x)$, which is well-defined, linear and bijective thanks to what we said above. Let us prove that F is an isometry. For all $x \in X$, we have that

$$\|F([x])\| = \|\pi_{Y^\perp}(x)\| = \|x - \pi_Y(x)\| = \inf_{y\in Y}\|x, y\| = \|[x]\|_{X/Y},$$

where we used the properties of the orthogonal projection and the definition of norm on the quotient space. By linearity, this is sufficient to prove that F is an isometry, as desired.

Solution of 5.2:

(i) We claim that Y is not closed. Let us consider the sequence $\{y_k\}_{k\geq 1}$ in Y given by $y_k := \sum_{i=1}^k e_{2i-1}/2^i$. Note that $y_k \in E_k \subset Y$ and the sequence converges to $\overline{y} := \sum_{i\geq 1} e_{2i-1}/2^i \in X$. However observe that $\overline{y}$ is not contained in Y. Indeed assume by contradiction that $\overline{y}$ is contained in Y, then there exists $k \geq 1$ such that $\overline{y} \in E_k$. However $\overline{y}$ cannot be contained in the span of $e_1, e_3, \dots, e_{2k-1}$, because otherwise we would find a nontrivial linear dependence for $e_1, e_3, e_5, \dots$, which is impossible since $\{e_i\}_{i\geq 1}$ is a Hilbertian basis.

(ii) Let us consider the closed subspace $Z := \overline{Y}$. Note that Z is a proper subspace since, for example, $e_2 \notin Z$, because e_2 is orthogonal to Y and thus to Z. Moreover observe that $\{e_{2i-1}\}_{i\geq 1}$ is a Hilbertian basis for Z. Let us consider the map $T\colon Z \to X$ defined on the basis $\{e_{2i-1}\}_{i\geq 1}$ as $T(e_{2i-1}) = e_i$. Note that T is easily bounded on the closed unit ball of Z, hence continuous. Indeed, for all $z = \sum_{i\geq 1} a_i e_{2i-1}$ with norm less or equal than 1 (i.e., $\|z\|^2 = \|\sum_{i\geq 1} a_i e_{2i-1}\|^2 = \sum_{i\geq 1}|a_i|^2 \leq 1$), we have that

$$\|T(z)\|^2 = \left\| T\left(\sum_{i\geq 1} a_i e_{2i-1}\right)\right\|^2 = \left\|\sum_{i\geq 1} a_i e_i\right\|^2 = \sum_{i\geq 1}|a_i|^2 = \left\|\sum_{i\geq 1} a_i e_{2i-1}\right\|^2 = \|z\|^2 \leq 1.$$

Moreover T is obviously a linear bijection. The inverse of T is defined on the basis $\{e_i\}_{i\geq 1}$ as $T^{-1}(e_i) = e_{2i-1}$. With the same computation as above, one can check that $\|T^{-1}(x)\| = \|x\|$ for all $x \in X$, which proves that T is a continuous linear bijection with continuous inverse, hence it is an isomorphism between Z and X.

Solution of 5.3:

(i) Let us consider the map $\phi\colon H \to H$ defined as $\phi(h)(x) := (h(x) + h(-x))/2$ for all $h \in H$ and $x \in (-1, 1)$. Note that

$$\|\phi(h)\|_{L^2} = \left\|\frac{h(x) + h(-x)}{2}\right\|_{L^2} \leq \frac{\|h(x)\|_{L^2} + \|h(-x)\|_{L^2}}{2} = \|h\|_{L^2},$$

thus ϕ is continuous. Moreover note that $h \in \ker \phi$ if and only if $h(x) + h(-x) = 0$ for a.e. $x \in (-1, 1)$, if and only if $h \in D$. This prove that $D = \ker \phi$ is a closed subspace of H, since it is the kernel of a continuous linear operator.

Analogously we can prove that $P = \ker \varphi$, where $\varphi \colon H \to H$ is the continuous linear operator given by $\varphi(h)(x) = (h(x) - h(-x))/2$ for all $h \in H$ and $x \in (-1, 1)$. Hence P is closed subspace of H as well.

(ii) Let us first prove that D and P are orthogonal. Consider $f \in D$ and $g \in P$, then we have that

$$\begin{aligned}\langle f, g\rangle_{L^2} &= \int_{-1}^{1} f(x)g(x) = \int_{0}^{1} f(x)g(x) + \int_{-1}^{0} f(x)g(x)\\ &= \int_{0}^{1} f(x)g(x) + \int_{0}^{1} f(-x)g(-x) = \int_{0}^{1} f(x)g(x) - \int_{0}^{1} f(x)g(x) = 0.\end{aligned}$$

Now let us consider any function $h \in H$ and define the functions $f(x) := \varphi(h)(x) = (h(x) - h(-x))/2$ and $g(x) := \phi(h)(x) = (h(x) + h(-x))/2$. Note that $f \in D$ and $g \in P$ and that

$$f(x) + g(x) = \frac{h(x) - h(-x)}{2} + \frac{h(x) + h(-x)}{2} = h(x).$$

This proves that $H = D \oplus P$, which, together with the orthogonality between D and P, proves that $D^\perp = P$ and $P^\perp = D$.

(iii) We claim that $\pi_D = \varphi$ and $\pi_P = \phi$. Let us prove the first equality, since the second one follows analogously. We have to check that $\langle h - \varphi(h), f\rangle = 0$ for all $f \in D$. However note that we proved in (ii) that $h - \varphi(h) = \phi(h) \in P$ and that $P \perp D$, hence $\langle h - \varphi(h), f\rangle = \langle \phi(f), f\rangle = 0$ for all $f \in D$, as desired.

(iv) We claim that $\mathcal{B}_D := \{\sin(\pi n x)\}_{n\geq 1}$ is a Hilbertian basis for D and that $\mathcal{B}_P := \{1/2\} \cup \{\cos(\pi n x)\}_{n\geq 1}$ is a Hilbertian basis for P. Indeed recall that $\mathcal{B}_D \cup \mathcal{B}_P$ is a Hilbertian basis for $L^2((-1, 1); \mathbb{R})$ and obviously we have that $\mathcal{B}_D \subset D$ and $\mathcal{B}_P \subset P$. This two fact together are easily sufficient to prove that $\mathcal{B}_D$ and $\mathcal{B}_P$ are Hilbert bases of D and P respectively.

Solution of 5.4: Recall that a Hilbertian basis of $L^2((-\pi, \pi); \mathbb{R})$ is

$$\{1/\sqrt{2\pi}\} \cup \{\cos(kx)/\sqrt{\pi}\}_{k\geq 1} \cup \{\sin(kx)/\sqrt{\pi}\}_{k\geq 1}.$$

Since $f(x) = x$ is odd, it can be express only in terms of the elements $\{\sin(kx)/\sqrt{\pi}\}_{k\geq 1}$ of the basis (see also (iv) in Problem 5.3). In particular we have that

$$f(x) = \sum_{k\geq 1} a_k \sin(kx),$$

where

$$a_k := \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(kx)x = \frac{1}{\pi}\left[-\frac{1}{k}\cos(kx)x\right]_{-\pi}^{\pi} + \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{k}\cos(kx) = \frac{2(-1)^{k+1}}{k}.$$

Hence, specifying Parseval's identity, we get that

$$\frac{2\pi^2}{3} = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 = \frac{1}{\pi}\|x\|_{L^2}^2 = \sum_{k\geq 1}|a_k|^2 = \sum_{k\geq 1}\frac{4}{k^2},$$

which implies exactly the desired formula by rearranging the terms.

Solution of 5.5: Since the subspace $V \subset X$ is closed in both statements (i) and (ii), the canonical quotient map $\pi\colon X \to X/V$ is continuous (Satz 2.3.1).

(i) $\dim \pi(U) \leq \dim U < \infty$ implies that $\pi(U) \subset X/V$ is closed (Satz 2.1.3). Since π is continuous, $\pi^{-1}(\pi(U)) = U + V \subset X$ is also closed.

(ii) Since $\dim \pi(U) \leq \dim(X/V) < \infty$, we can argue the same way as in (i).

Solution of 5.6:

(i) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in U which converges to f in $(X, \|\cdot\|_X)$. Then, since $f_n(0) = 0 = f_n(1)$, we can colclude $f(0) = 0 = f(1)$, i.e., $f \in U$ by passing to the limit $n \to \infty$ in the following inequalities.

$$|f(0)| = |f_n(0) - f(0)| \leq \sup_{x\in[0,1]}|f_n(x) - f(x)| = \|f_n - f\|_X,$$
$$|f(1)| = |f_n(1) - f(1)| \leq \sup_{x\in[0,1]}|f_n(x) - f(x)| = \|f_n - f\|_X.$$

(ii) Let $u_1, u_2 \in X$ be given by $u_1(t) = 1 - t$ and $u_2(t) = t$. We claim that the equivalence classes $[u_1], [u_2] \in X/U$ form a basis for X/U.

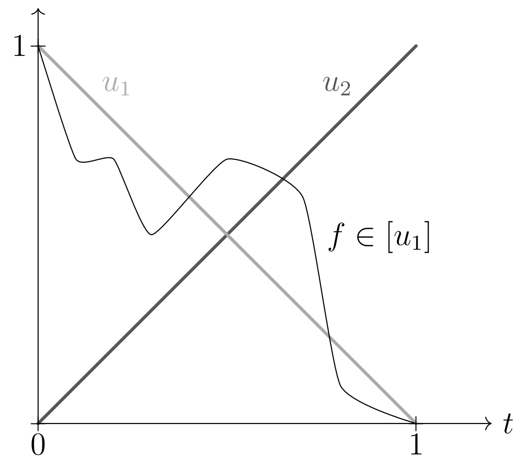

Figure 1: The functions $u_1, u_2 \in X$ and some $f \in [u_1]$.

To prove linear independence, let $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\lambda_1[u_1] + \lambda_2[u_2] = 0 \in X/U$ which means $\lambda_1 u_1 + \lambda_2 u_2 \in U$. This implies by definition

$$\lambda_1 = \lambda_1 u_1(0) + \lambda_2 u_2(0) = 0 = \lambda_1 u_1(1) + \lambda_2 u_2(1) = \lambda_2.$$

To show that $[u_1]$ and $[u_2]$ span X/U, let $[h] \in X/U$ with representative $h \in X$. By evaluation at $t = 0$ and $t = 1$, we conclude

$$\Big(t \mapsto h(t) - h(0)u_1(t) - h(1)u_2(t)\Big) \in U.$$

This implies $[h] = h(0)[u_1] + h(1)[u_2]$ in X/U which proves the claim.

Remark. The components of $[h]$ in this basis are unique since every representative $\tilde{h} \in [h]$ must have the same boundary values $\tilde{h}(0) = h(0)$ and $\tilde{h}(1) = h(1)$.