6.1. 1D "closed graph theorem" for continuous functions ✍. Let $f\colon \mathbb{R} \to \mathbb{R}$ be bounded and let $\Gamma = \{(x,y) \in \mathbb{R}^2 \mid y = f(x)\} \subset \mathbb{R}^2$ be its graph. Show that f is continuous if and only if Γ is closed. (Achtung: the point here is that we are not restricting to linear maps.)

Is the same statement true if f is *not* assumed to be bounded?

6.2. Closed range ✍. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces. Let $A\colon D_A \subset X \to Y$ be a linear operator with closed graph. Show that the following statements are equivalent.

(i) A is injective and its range $W_A := A(D_A)$ is closed in $(Y, \|\cdot\|_Y)$.

(ii) $\exists C > 0 \quad \forall x \in D_A: \quad \|x\|_X \leq C\|Ax\|_Y$.

Hint. One implication follows from the Inverse Mapping Theorem.

6.3. An implication of Hellinger–Töplitz (coercive operators) ✍. Let $(H, (\cdot,\cdot))$ be a Hilbert space and let $A\colon H \to H$ be a symmetric linear operator such that

$$\exists \lambda > 0 \quad \forall x \in H: \quad (Ax, x) \geq \lambda \|x\|^2.$$

(Any linear operator satisfying such an inequality is called *coercive*.) Show that A is an isomorphism of normed spaces and $\|A^{-1}\| \leq \lambda^{-1}$.

6.4. Graph norm ⚙.

(i) Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and let $A\colon D_A \subset X \to Y$ be a linear operator with graph $\Gamma_A \subset X \times Y$. Then, $\|x\|_{\Gamma_A} := \|x\|_X + \|Ax\|_Y$ defined on D_A is called the *graph norm*.

Show that if A has closed graph, then $(D_A, \|\cdot\|_{\Gamma_A})$ is a Banach space.

(ii) Let $(X_0, \|\cdot\|_{X_0})$, $(X_1, \|\cdot\|_{X_1})$ and $(X_2, \|\cdot\|_{X_2})$ be Banach spaces and let

$$T_1\colon D_1 \subset X_0 \to X_1,$$
$$T_2\colon D_2 \subset X_0 \to X_2$$

be linear operators with closed graphs such that $D_1 \subset D_2$. Prove that

$$\exists C > 0 \quad \forall x \in D_1: \quad \|T_2 x\|_{X_2} \leq C\Big(\|T_1 x\|_{X_1} + \|x\|_{X_0}\Big).$$

6.5. Closed sum ⚙. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and let

$$A\colon D_A \subset X \to Y,$$
$$B\colon D_B \subset X \to Y$$

be linear operators with $D_A \subset D_B$. Under the assumption that there exist constants $0 \leq a < 1$ and $b \geq 0$ such that

$$\forall x \in D_A\colon \quad \|Bx\|_Y \leq a\|Ax\|_Y + b\|x\|_X,$$

show that if A has closed graph, then $(A+B)\colon D_A \to Y$ has closed graph.

Hint. Given a sequence $(x_n)_{n\in\mathbb{N}}$ in D_A, prove the estimate

$$(1-a)\|A(x_n - x_m)\| \leq \|(A+B)(x_n - x_m)\| + b\|x_n - x_m\|.$$

6.6. Derivative operator ⚙. Let $X = L^2([0,1])$. On $D_A := C_c^\infty((0,1)) \subset X$ we define the derivative operator

$$\begin{aligned} A\colon D_A &\to X \\ f &\mapsto f'. \end{aligned}$$

Recall that A is closable. Show that the domain $D_{\overline{A}}$ of its closure is contained in

$$\{f \in C^0([0,1]) \mid f(0) = 0 = f(1)\}.$$

Hint. Given $f \in D_{\overline{A}}$ consider a sequence $(f_n)_{n\in\mathbb{N}}$ in D_A which converges to f in X. *Achtung*, L^2-convergence does *not* imply pointwise convergence: You cannot evaluate f at points. Instead, compare $f_n(t)$ to $g(t) := \int_0^t \overline{A}f \,\mathrm{d}x$.

6.7. Closable inverse ✎. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces. Let $A\colon D_A \subset X \to Y$ be a closable linear operator. Assume that its closure $\overline{A}$ is injective. Show that the inverse operator A^{-1} is closable and $\overline{A^{-1}} = (\overline{A})^{-1}$.

Hint. Consider the image of the graph of A under the map

$$\begin{aligned} \chi\colon X \times Y &\to Y \times X \\ (x,y) &\mapsto (y,x). \end{aligned}$$

6.8. Derivative operator on different spaces ✎. For the Banach spaces

(i) $X = Y = C^0([0,1])$ with norm $\|\cdot\|_{C^0([0,1])}$

(ii) $X = Y = L^2([0,1])$ with norm $\|\cdot\|_{L^2([0,1])}$

of functions $f\colon [0,1] \to \mathbb{R}$, $t \mapsto f(t)$, we consider the linear operator

$$\frac{\mathrm{d}}{\mathrm{d}t}\colon C^1([0,1]) \subset X \to Y.$$

In both cases, discuss whether this operator is bounded and whether it is closable.

Remark. Given normed spaces X and Y, a linear map $A\colon X \to Y$ is continuous or, equivalently, bounded if it satisfies one (hence all) of the conditions given in Satz 2.2.1.

6. Solutions

Solution of 6.1: Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function with graph $\Gamma \subset \mathbb{R}^2$. Let $((x_n, y_n))_{n\in\mathbb{N}}$ be a sequence in Γ converging to some $(x, y) \in \mathbb{R}^2$. Then, continuity of f implies

$$f(x) = f\Big(\lim_{n\to\infty} x_n\Big) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} y_n = y$$

which means that $(x, y) \in \Gamma$. Therefore, Γ is closed.

Conversely, let $f\colon \mathbb{R} \to \mathbb{R}$ be a bounded function with closed graph Γ. Given $x \in \mathbb{R}$, let $(x_n)_{n\in\mathbb{N}}$ be any sequence in $\mathbb{R}$ with limit x. Let $y_n = f(x_n)$. Since f is bounded, the sequence $(y_n)_{n\in\mathbb{N}}$ is bounded and therefore has a convergent subsequence $(y_{n_k})_{k\in\mathbb{N}}$. Let $y \in \mathbb{R}$ be its limit. Since Γ is closed, $(x, y) = \lim_{k\to\infty}(x_{n_k}, y_{n_k}) \in \Gamma$ which means that $f(x) = y$. Therefore, $y = f(x)$ is the unique limit of any convergent subsequence of $(y_n)_{n\in\mathbb{N}}$. Since $(y_n)_{n\in\mathbb{N}}$ is bounded with only one accumulation point $y \in \mathbb{R}$ we have $y_n \to y$ as $n \to \infty$. This implies continuity of f at x by virtue of

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} y_n = y = f(x) = f\Big(\lim_{n\to\infty} x_n\Big).$$

If f is *not* assumed to be bounded, we can construct the following counterexample.

$$f(x) = \begin{cases} \frac{1}{x} & \text{for } x > 0, \\ 0 & \text{for } x \leq 0. \end{cases}$$

Since both restrictions $f|_{(0,\infty)}$ and $f|_{(-\infty,0]}$ are continuous, both connected components of the graph of f are closed but f is not continuous at $x = 0$.

Solution of 6.2:

"(i) ⇒ (ii)" As a closed subspace of a complete space, $(W_A, \|\cdot\|_Y)$ is complete. Since $A\colon D_A \subset X \to W_A$ is bijective with closed graph and since X, W_A are Banach spaces, we may apply the Inverse Mapping Theorem to obtain a continuous inverse $A^{-1}\colon W_A \to D_A$. In particular, $\|A^{-1}\| =: C$ is finite and for every $x \in D_A$ we have

$$\|x\|_X = \|A^{-1}Ax\|_X \leq \|A^{-1}\|\|Ax\|_Y = C\|Ax\|_Y.$$

"(ii) ⇒ (i)" Let $x \in D_A$ with $Ax = 0$. Then, (ii) implies $\|x\|_X \leq 0$, hence $x = 0$. This implies that the linear map A is injective.

Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in W_A converging to some $y \in Y$. By definition of W_A there exist $x_n \in D_A$ such that $Ax_n = y_n$. For every $n, m \in \mathbb{N}$, assumption (ii) implies

$$\|x_n - x_m\|_X \leq C\|Ax_n - Ax_m\|_Y = C\|y_n - y_m\|_Y.$$

From $(y_n)_{n\in\mathbb{N}}$ being Cauchy in $(Y, \|\cdot\|_Y)$, we conclude that $(x_n)_{n\in\mathbb{N}}$ is Cauchy in $(X, \|\cdot\|_X)$. Since $(X, \|\cdot\|_X)$ is complete, there exists $X \ni x = \lim_{n\to\infty} x_n$. Since the graph of A is assumed to be closed, $x \in D_A$ and $Ax = y$. Therefore, $y \in W_A$ and we conclude that W_A is a closed subspace of Y.

Solution of 6.3: First note that, by Hellinger–Töplitz theorem (Beispiel 3.3.2), A is continuous. Moreover observe that the coercivity condition on A implies that

$$\|Ax\|\|x\| \geq (Ax, x) \geq \lambda\|x\|^2 \implies \|Ax\| \geq \lambda\|x\|. \tag{1}$$

In particular A is a linear operator, which is continuous (hence with closed graph), and such that (ii) in Problem 6.2 holds. Hence, by the equivalence between (i) and (ii) in Problem 6.2, we obtain that A is injective and its range $W_A := A(H)$ is closed in H, namely $\overline{W_A} = W_A$.

To prove that A is surjective, let $x \in W_A^\perp$. Then

$$0 = (Ax, x) \geq \lambda\|x\|^2$$

which implies $x = 0$. Therefore, $W_A^\perp = \{0\}$ and thus $W_A = \overline{W_A} = (W_A^\perp)^\perp = H$, which proves the surjectivity of A.

Hence we have shown that A is a continuous, bijective linear operator. The Inverse Mapping Theorem already implies that A has a continuous inverse. Finally, thanks to (1), we have that

$$\|A^{-1}x\| \leq \frac{1}{\lambda}\|AA^{-1}x\| = \frac{\|x\|}{\lambda},$$

which implies $\|A^{-1}\| \leq \lambda^{-1}$ and thus concludes the proof.

Solution of 6.4:

(i) Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(D_A, \|\cdot\|_{\Gamma_A})$. The definition of graph norm implies that $(x_n)_{n\in\mathbb{N}}$ is Cauchy in $(X, \|\cdot\|_X)$ and $(Ax_n)_{n\in\mathbb{N}}$ is Cauchy in $(Y, \|\cdot\|_Y)$. Since both spaces are complete, there exist $x \in X$ and $y \in Y$ such that $\|x_n - x\|_X \to 0$ and $\|Ax_n - y\|_Y \to 0$ as $n \to \infty$. Since the graph of A is closed, we have $x \in D_A$ with $Ax = y$. Therefore, $\|x - x_n\|_{\Gamma_A} \to 0$ which proves that $(D_A, \|\cdot\|_{\Gamma_A})$ is complete.

(ii) Let Γ_1 and Γ_2 be the graphs of T_1 and T_2 respectively. Since T_1 and T_2 have closed graphs by assumption, (i) implies that $(D_1, \|\cdot\|_{\Gamma_1})$ and $(D_2, \|\cdot\|_{\Gamma_2})$ are Banach spaces. Since $D_1 \subset D_2$, we can consider the identity map $\mathrm{id}\colon (D_1, \|\cdot\|_{\Gamma_1}) \to (D_2, \|\cdot\|_{\Gamma_2})$ and claim that its graph is closed. Indeed, assume that $x_n \to x$ in $(D_1, \|\cdot\|_{\Gamma_1})$ and $\mathrm{id}(x_n) = x_n \to y$ in $(D_2, \|\cdot\|_{\Gamma_2})$. Then, the definition of graph norm implies that both, $\|x_n - x\|_{X_0} \to 0$ and $\|x_n - y\|_{X_0} \to 0$ as $n \to \infty$ which implies $x = y$ and proves the claim. The closed graph theorem implies that id is continuous, which means

$$\exists C > 0 \quad \forall x \in D_1 : \quad \|x\|_{\Gamma_2} \leq C\|x\|_{\Gamma_1}.$$

By definition, this implies $\|T_2x\|_{X_2} \leq C\Big(\|T_1x\|_{X_1} + \|x\|_{X_0}\Big) - \|x\|_{X_0}$.

Solution of 6.5: Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in D_A. Then, by triangle inequality and the assumption,

$$\begin{aligned}\|A(x_n - x_m)\|_Y - \|(A+B)(x_n - x_m)\|_Y &\leq \|B(x_n - x_m)\|_Y \\ &\leq a\|A(x_n - x_m)\|_Y + b\|x_n - x_m\|_X.\end{aligned}$$

which implies the estimate

$$(1-a)\|A(x_n-x_m)\|_Y \leq \|(A+B)(x_n-x_m)\|_Y + b\|x_n-x_m\|_X. \tag{$\dagger$}$$

Assume that $x_n \to x$ in X and $(A+B)x_n \to y$ in Y. The claim is $(A+B)x = y$. Since $a<1$, estimate ($\dagger$) implies that $(Ax_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(Y, \|\cdot\|_Y)$ and therefore convergent to some $\tilde{y} \in Y$. Since the graph of A is a closed by assumption, we have $x \in D_A$ with $Ax = \tilde{y}$. Therefore, we may conclude

$$\|B(x-x_n)\|_Y \leq a\|A(x-x_n)\| + b\|x-x_n\| \xrightarrow{n\to\infty} 0$$

which implies $Bx_n \to Bx$ in Y and thus,

$$y = \lim_{n\to\infty}(A+B)x_n = \lim_{n\to\infty}Ax_n + \lim_{n\to\infty}Bx_n = Ax + Bx = (A+B)x.$$

Solution of 6.6: Let $f \in D_{\overline{A}}$. Then there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in D_A with

$$\|f_n - f\|_{L^2([0,1])} \to 0, \qquad\qquad \|Af_n - \overline{A}f\|_{L^2([0,1])} \to 0$$

as $n \to \infty$. By definition, $f_n \in C_c^\infty((0,1)) \subset L^2([0,1])$ has a representative which (after extension to $t=0$ and $t=1$) is a smooth function $\tilde{f}_n \colon [0,1] \to \mathbb{R}$ with $\tilde{f}_n(0) = 0 = \tilde{f}_n(1)$. But L^2-convergence alone is *not* enough to conclude the same for f. Instead, we observe, that $\tilde{f}_n$ is a primitive of $A\tilde{f}_n = (\tilde{f}_n)'$. We hope that f arises as a primitive of $\overline{A}f \in L^2([0,1])$. Therefore, we consider the function $g\colon [0,1] \to \mathbb{R}$ given by

$$g(t) := \int_0^t \overline{A}f \,\mathrm{d}x.$$

We apply Hölder's inequality to estimate

$$\begin{aligned}\left|\tilde{f}_n(t) - g(t)\right| &= \left|\int_0^t \left(\tilde{f}_n' - \overline{A}f\right)\mathrm{d}x\right| \leq \int_0^t \left|A\tilde{f}_n - \overline{A}f\right|\mathrm{d}x \\ &\leq \left(\int_0^t 1\,\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_0^t \left|A\tilde{f}_n - \overline{A}f\right|^2\mathrm{d}x\right)^{\frac{1}{2}} \leq \left\|A\tilde{f}_n - \overline{A}f\right\|_{L^2([0,1])} \xrightarrow{n\to\infty} 0.\end{aligned}$$

Taking the supremum over $t \in [0,1]$, we see that $(\tilde{f}_n)_{n\in\mathbb{N}}$ converges *uniformly* to g. Hence g is continuous, as uniform limit of continuous functions. As a result, since uniform convergence implies L^2-convergence, g must be a continuous representative of the same L^2-class of f, being f the L^2-limit of $(\tilde{f}_n)_{n\in\mathbb{N}}$. Finally, uniform convergence implies pointwise convergence, in particular

$$g(0) = \lim_{n\to\infty}\tilde{f}_n(0) = 0, \qquad\qquad g(1) = \lim_{n\to\infty}\tilde{f}_n(1) = 0.$$

Solution of 6.7: Since the closure $\overline{A}$ is assumed to be injective, A is injective and therefore has an inverse $A^{-1}\colon W_A \to D_A$, where $W_A := A(D_A)$ denotes the range of A. Defining

$$\begin{aligned}\chi\colon X\times Y &\to Y \times X \\ (x,y) &\mapsto (y,x)\end{aligned}$$

we observe that the graph $\Gamma_{A^{-1}}$ of A^{-1} is given by

$$\Gamma_{A^{-1}} := \{(y,x) \in Y \times X \mid y \in W_A,\ x = A^{-1}y\} = \chi(\Gamma_A).$$

Since χ is an isomorphism of normed spaces, we have

$$\overline{\Gamma_{A^{-1}}} = \overline{\chi(\Gamma_A)} = \chi(\overline{\Gamma_A}) = \chi(\Gamma_{\overline{A}}) = \Gamma_{(\overline{A})^{-1}}$$

This proves that $\overline{\Gamma_{A^{-1}}}$ is the graph of the linear operator $(\overline{A})^{-1}$ (which is well-defined, since $\overline{A}$ is injective). Therefore, A^{-1} is closable as claimed and

$$\Gamma_{\overline{A^{-1}}} = \overline{\Gamma_{A^{-1}}} = \Gamma_{(\overline{A})^{-1}} \qquad \Longrightarrow \quad \overline{A^{-1}} = (\overline{A})^{-1}.$$

Solution of 6.8:

(i) The operator $\frac{\mathrm{d}}{\mathrm{d}t}\colon C^1([0,1]) \subset C^0([0,1]) \to C^0([0,1])$ is not bounded. A counterexample are the functions $f_n \in C^1([0,1])$ for $n \in \mathbb{N}$ given by $f_n(t) = t^n$. Indeed,

$$\left.\begin{aligned} \|f_n\|_{C^0([0,1])} &= \max_{t\in[0,1]} t^n = 1, \\ \|\tfrac{\mathrm{d}}{\mathrm{d}t} f_n\|_{C^0([0,1])} &= \max_{t\in[0,1]} nt^{n-1} = n, \end{aligned}\right. \qquad \Longrightarrow \qquad \frac{\|\frac{\mathrm{d}}{\mathrm{d}t} f_n\|_{C^0([0,1])}}{\|f_n\|_{C^0([0,1])}} = n.$$

To check, whether the operator is closable, we consider a sequence $(u_k)_{k\in\mathbb{N}}$ of functions $u_k \in C^1([0,1])$ such that $\|u_k\|_{C^0([0,1])} \to 0$ as $k \to \infty$. Suppose, $v \in C^0([0,1])$ is a limit of $v_k := \frac{\mathrm{d}}{\mathrm{d}t} u_k$ in the sense that $\|v - v_k\|_{C^0([0,1])} \to 0$. Does $v = 0$ follow? Yes, in fact, for any $\varphi \in C_c^\infty((0,1))$, integration by parts yields (the boundary terms vanish due to $\varphi(0) = 0 = \varphi(1)$)

$$\left|\int_0^1 v_k(t)\varphi(t)\,\mathrm{d}t\right| = \left|-\int_0^1 u_k(t)\varphi'(t)\,\mathrm{d}t\right| \leq \left(\int_0^1 |\varphi'(t)|\,\mathrm{d}t\right)\|u_k\|_{C^0([0,1])} \xrightarrow{k\to\infty} 0.$$

Since $\|v - v_k\|_{C^0([0,1])} \to 0$ implies

$$\int_0^1 v(t)\varphi(t)\,\mathrm{d}t = \lim_{k\to\infty} \int_0^1 v_k(t)\varphi(t)\,\mathrm{d}t = 0$$

and since $\varphi \in C_c^\infty((0,1))$ is arbitrary, we have $v(t) = 0$ for almost every $t \in [0,1]$. As v is continuous, this implies $v \equiv 0$ on $[0,1]$. Therefore, the operator is closable.

(ii) The operator $\frac{\mathrm{d}}{\mathrm{d}t}\colon C^1([0,1]) \subset L^2([0,1]) \to L^2([0,1])$ is not bounded. A counterexample are the functions $g_n \in C^1([0,1])$ for $n \in \mathbb{N}$ given by $g_n(t) = e^{nt}$. Indeed,

$$\left.\begin{aligned} \|g_n\|_{L^2([0,1])} &= \left(\int_0^1 e^{2nt}\,\mathrm{d}t\right)^{\frac12} = \frac{1}{\sqrt{2n}}(e^{2n}-1)^{1/2}, \\ \|\tfrac{\mathrm{d}}{\mathrm{d}t} g_n\|_{L^2([0,1])} &= \left(\int_0^1 (ne^{nt})^2\,\mathrm{d}t\right)^{\frac12} = \frac{n}{\sqrt{2n}}(e^{2n}-1)^{1/2}, \end{aligned}\right. \qquad \Longrightarrow \qquad \frac{\|\frac{\mathrm{d}}{\mathrm{d}t} g_n\|_{L^2([0,1])}}{\|g_n\|_{L^2([0,1])}} = n.$$

To check, whether the operator is closable, we consider a sequence $(u_k)_{k\in\mathbb{N}}$ of functions $u_k \in C^1([0,1])$ such that $\|u_k\|_{L^2([0,1])} \to 0$ as $k \to \infty$. Suppose, $v \in L^2([0,1])$ is a limit of

$v_k := \frac{\mathrm{d}}{\mathrm{d}t} u_k$ in the sense that $\|v - v_k\|_{L^2([0,1])} \to 0$. Does $v = 0$ follow? Yes, in fact, for any $\varphi \in C_c^\infty((0,1))$ using Hölder's inequality, we have

$$\left|\int_0^1 v_k(t)\varphi(t)\,\mathrm{d}t\right| = \left|-\int_0^1 u_k(t)\varphi'(t)\,\mathrm{d}t\right| \leq \|u_k\|_{L^2([0,1])}\|\varphi'\|_{L^2([0,1])} \xrightarrow{n\to\infty} 0.$$

Since $\|v - v_k\|_{L^2([0,1])} \to 0$ implies (for instance by continuity of the L^2-scalar product)

$$\int_0^1 v(t)\varphi(t)\,\mathrm{d}t = \lim_{k\to\infty} \int_0^1 v_k(t)\varphi(t)\,\mathrm{d}t = 0$$

and since $\varphi \in C_c^\infty((0,1))$ is arbitrary, we have $v = 0$ in $L^2([0,1])$. (For that we do not care about the values at $t = 0$ or $t = 1$.) Therefore, the operator is closable.