7.1. Dense kernel ✍. Let $(X, \|\cdot\|_X)$ be a normed space and let $f\colon X \to \mathbb{R}$ be linear and not identically zero. Show that f is *not* continuous if and only if $\ker(f)$ is dense in X.

7.2. Complementing subspaces of finite dimension or codimension ⚙. Let $(X, \|\cdot\|_X)$ be a Banach space and $U \subset X$ a closed subspace. Prove that:

(i) if $\dim(U) = n < \infty$, then U is topologically complemented;

(ii) if $\dim(X/U) = m < \infty$, then U is topologically complemented.

Remark. The notion of topological complement was defined in Problem 3.4. There we showed that $U \subset X$ is topologically complemented if (and only if) there exists a continuous linear map $P\colon X \to X$ with $P \circ P = P$ and image $P(X) = U$.

7.3. Attaining the distance from the kernel ⚙ 💎. Let $(X, \|\cdot\|_X)$ be a normed space and let $\varphi\colon X \to \mathbb{R}$ be a continuous linear functional. Assume $N := \ker(\varphi) \subsetneq X$ and let $x_0 \in X \setminus N$. Prove that the following statements are equivalent.

(i) There exists $y_0 \in N$ with $\|x_0 - y_0\|_X = \operatorname{dist}(x_0, N)$.

(ii) There exists $x_1 \in X$ with $\|x_1\|_X = 1$ and $\|\varphi\| = |\varphi(x_1)|$.

7.4. Not attaining the distance from the kernel ✍. Consider the Banach space $(X, \|\cdot\|_X) = \big(C^0([-1,1]), \|\cdot\|_{C^0([-1,1])}\big)$. Consider the map $\varphi\colon X \to \mathbb{R}$ given by

$$\varphi(f) = \int_0^1 f(t)\,\mathrm{d}t - \int_{-1}^0 f(t)\,\mathrm{d}t.$$

(i) Show that $\varphi \in L(X, \mathbb{R})$ with $\|\varphi\|_{L(X,\mathbb{R})} = 2$.

(ii) Prove that there does not exist $f \in X$ with $\|f\|_X = 1$ and $|\varphi(f)| = 2$.

Now consider the kernel $N := \{f \in X \mid \varphi(f) = 0\}$ of φ and some $g_0 \in X \setminus N$.

(iii) Show that N is closed.

(iv) Show that there does not exist any $f_0 \in N$ such that $\|f_0 - g_0\|_X = \operatorname{dist}(g_0, N)$.

7.5. Unique extension of functionals on Hilbert spaces ⚙. Let $(H, (\cdot,\cdot)_H))$ be a Hilbert space. Let $Y \subset H$ be any subspace and let $f\colon Y \to \mathbb{R}$ be a continuous linear functional. The Hahn-Banach Theorem allows an extension $F\colon H \to \mathbb{R}$ with $F|_Y = f$ and $\|F\| = \|f\|$. Prove that F is unique.

7.6. Distance from convex sets in Hilbert spaces ✍. Let $(H, (\cdot,\cdot)_H)$ be a Hilbert space and $\emptyset \neq Q \subset H$ a convex subset. Let $x \in H$ with distance $d := \operatorname{dist}(x, Q)$ from Q. Prove the following statements.

(i) Every sequence $(x_n)_{n\in\mathbb{N}}$ in Q with $\lim\limits_{n\to\infty} \|x_n - x\|_H = d$ is a Cauchy sequence in H.

(ii) Suppose Q is closed in H. Then there exists a unique $y \in Q$ with $\|x - y\|_H = d$.

7. Solutions

Solution of 7.1:

"⇒" Suppose that f is not continuous. Then there exists a sequence $(x_k)_{k\in\mathbb{N}}$ in X, which can be normed to $\|x_k\|_X = 1$ by linearity of f, such that $|f(x_k)| \to \infty$ as $k \to \infty$. Without loss of generality, we can assume $f(x_k) \neq 0$ for every $k \in \mathbb{N}$. The goal is to approximate any $z \in X$ by elements $y_k \in \ker(f)$. For each $k \in \mathbb{N}$ we define

$$y_k := z - \frac{f(z)}{f(x_k)} x_k, \qquad \Longrightarrow \quad f(y_k) = f(z) - \frac{f(z)}{f(x_k)} f(x_k) = 0 \qquad \Longrightarrow \quad y_k \in \ker(f).$$

Indeed, the sequence $(y_k)_{k\in\mathbb{N}}$ approximates z in X because

$$\|z - y_k\|_X = \left|\frac{f(z)}{f(x_k)}\right| \|x_k\|_X = \frac{|f(z)|}{|f(x_k)|} \xrightarrow{k\to\infty} 0'$$

Hence we have shown that $\ker(f)$ is dense in X.

"⇐" Conversely, we assume $\overline{\ker(f)} = X$ and claim that f is not continuous. Since we assume $f \not\equiv 0$ there exists $x \in X$ with $f(x) \neq 0$. Since the kernel is dense, there exists a sequence $(x_k)_{k\in\mathbb{N}}$ in $\ker(f)$ with $\|x_k - x\|_X \to 0$ as $k \to \infty$. But this violates continuity, since $\lim\limits_{k\to\infty} f(x_k) = 0 \neq f(x)$.

Solution of 7.2:

(i) Let $e_1, \ldots, e_n$ be a basis of the given finite-dimensional subspace $U \subset X$. For $i \in \{1, \ldots, n\}$, we define the linear functionals $f_i \colon U \to \mathbb{R}$ by

$$f_i(e_j) = \delta_{ij} := \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{else.} \end{cases}$$

Recall that, by linearity, it suffices to define the functionals on a basis of U. Since U is finite dimensional, $f_i \in L(U;\mathbb{R})$. From the Hahn-Banach Theorem follows (Satz 4.1.3) that there exist extensions $F_i \in L(X;\mathbb{R})$ with $\|F_i\| = \|f_i\|$. We define

$$\begin{aligned} P \colon X &\to X \\ x &\mapsto \sum_{i=1}^{n} F_i(x)\, e_i. \end{aligned}$$

Then, P is linear and also continuous, since

$$\|Px\|_X \leq \Big(\sum_{i=1}^{n} \|F_i\| \|e_i\|_X\Big) \|x\|_X.$$

By construction, $P(X) \subset \operatorname{span}\{e_1, \ldots, e_n\} = U$. Moreover by definition of f_i and F_i we have $P(e_i) = e_i$ for every $i \in \{1, \ldots, n\}$. Therefore, $P(X) = U$. Finally, for every $x \in X$,

$$(P \circ P)(x) = P\Big(\sum_{i=1}^{n} F_i(x)\, e_i\Big) = \sum_{i=1}^{n} F_i(x)\, P(e_i) = \sum_{i=1}^{n} F_i(x)\, e_i = P(x).$$

From Problem 3.4 (i) then follows that U is topologically complemented.

(ii) Recall that the quotient space X/U consists of equivalence classes, which we denote by $[x]$ for every $x \in X$, and comes with a canonical quotient map $\pi\colon X \to X/U$. Since $\dim(X/U) = m < \infty$ we can choose a basis $[e_1], \dots, [e_m]$ for X/U along with a representative $e_i \in X$ for each element $[e_i]$ of the basis. As in (i) we define linear functionals $f_i\colon X/U \to \mathbb{R}$ for $i \in \{1, \dots, m\}$ by $f_i([e_j]) = \delta_{ij}$. Now, we just set $F_i := f_i \circ \pi\colon X \to \mathbb{R}$ in order to define

$$\begin{aligned} P\colon X &\to X \\ x &\mapsto \sum_{i=1}^{n} F_i(x)\, e_i. \end{aligned}$$

Since $F_i(e_j) = f_i(\pi(e_j)) = f_i([e_j]) = \delta_{ij}$ we have $P \circ P = P$ as in (i). Since $[e_1], \dots, [e_m]$ is a basis for X/U, the representatives $e_1, \dots, e_m$ must be linearly independent in X. Therefore, $P(x) = 0$ implies $F_i(x) = f_i([x]) = 0$ for every $i \in \{1, \dots, m\}$, which in turn implies $[x] = [0]$ or $x \in U$. Conversely, $x \in U$ implies $\pi(x) = [0]$ and $P(x) = 0$. Thus we have shown $\ker(P) = U$. As in Problem 3.4 (i), we conclude that $(1 - P)$ is a continuous projection onto U which implies that U is topologically complemented.

Solution of 7.3: Given a normed space $(X, \|\cdot\|_X)$, a continuous linear functional $\varphi\colon X \to \mathbb{R}$ with kernel $N := \ker(\varphi) \subsetneq X$ and a point $x_0 \in X \setminus N$, the claim is equivalence of

(i) There exists $y_0 \in N$ with $\|x_0 - y_0\|_X = \operatorname{dist}(x_0, N) =: d$,

(ii) There exists $x_1 \in X$ with $\|x_1\|_X = 1$ and $\|\varphi\| = |\varphi(x_1)|$.

The first isomorphism theorem states that the quotient space X/N is isomorphic to the image of φ. Since $\varphi(x_0) \neq 0$ and $\varphi(\lambda x_0) = \lambda\varphi(x_0)$ for every $\lambda \in \mathbb{R}$, the image of φ is $\mathbb{R}$, which means that X/N is one-dimensional. Therefore, every element $[x] \in X/N$ is of the form $[x] = t[x_0]$ for some uniquely determined $t \in \mathbb{R}$. This means that every element $x \in X$ is of the form $x = tx_0 + y$ for some uniquely determined $t \in \mathbb{R}$ and $y \in N$.

"(i) ⇒ (ii)" Let us denote $d := \|x_0 - y_0\|_X = \operatorname{dist}(x_0, N)$ and define $x_1 = (x_0 - y_0)/d$, then $\|x_1\|_X = 1$. We claim that $|\varphi(x_1)| = \|\varphi\|$. First of all we have

$$|\varphi(x_1)| = \frac{|\varphi(x_1)|}{\|x_1\|_X} \le \sup_{x \in X} \frac{|\varphi(x)|}{\|x\|_X} = \|\varphi\|.$$

Given any $x \in X$, let $t \in \mathbb{R}$ and $y \in N$ be as above. If $t = 0$, then $\varphi(x) = 0$. Therefore, we assume $t \neq 0$ and observe

$$\begin{aligned} |\varphi(x)| &= |\varphi(tx_0 + y)| = |\varphi(td\, x_1 + t\, y_0 + y)| = |t|d\, |\varphi(x_1)|, \\ \|x\|_X &= \|tx_0 + y\|_X = |t| \|x_0 + \tfrac{1}{t} y\|_X \ge |t| \inf_{\tilde{y} \in N} \|x_0 - \tilde{y}\|_X = |t| d. \end{aligned}$$

This implies that

$$|\varphi(x_1)| = \frac{|\varphi(x_1)|}{\|x_1\|_X} \le \sup_{x \in X} \frac{|\varphi(x)|}{\|x\|_X} \le \frac{|t|d\, |\varphi(x_1)|}{|t|d} = |\varphi(x_1)|.$$

Thus, the inequalities above are in fact equalities and we conclude $|\varphi(x_1)| = \|\varphi\|$.

"(ii) ⇒ (i)" As above, we have $x_1 = tx_0 + y_1$ for some uniquely determined $t \in \mathbb{R}$ and $y_1 \in N$. In fact, $t \neq 0$ since $|\varphi(x_1)| = \|\varphi\| \neq 0$. Therefore, $x_0 = \frac{1}{t}(x_1 - y_1)$ and

$$\|x_0 + \tfrac{1}{t}y_1\|_X = \|\tfrac{1}{t}x_1\|_X = \frac{1}{|t|}.$$

Now we use the fact that any $z \in X$ satisfies the estimate

$$\|\varphi\| \geq \frac{|\varphi(z)|}{\|z\|_X} \quad \Longrightarrow \quad \|z\|_X \geq \frac{|\varphi(z)|}{\|\varphi\|},$$

to obtain that, given any $y \in N$, we have

$$\begin{aligned}\|x_0 - y\|_X &= \|\tfrac{1}{t}x_1 - \tfrac{1}{t}y_1 - y\|_X \\ &\geq \frac{|\varphi(\frac{1}{t}x_1 - \frac{1}{t}y_1 - y)|}{\|\varphi\|} = \frac{|\varphi(x_1)|}{|t|\|\varphi\|} = \frac{1}{|t|} = \|x_0 + \tfrac{1}{t}y_1\|_X.\end{aligned}$$

Since $y_0 := -\frac{1}{t}y_1 \in N$ we conclude that y_0 attains $\operatorname{dist}(x_0, N)$.

Solution of 7.4:

(i) Linearity of $\varphi\colon X \to \mathbb{R}$ follows from linearity of the integral. Moreover,

$$|\varphi(f)| \leq \int_0^1 |f(t)|\,\mathrm{d}t + \int_{-1}^0 |f(t)|\,\mathrm{d}t \leq 2\|f\|_{C^0([-1,1])} = 2\|f\|_X$$

implies

$$\|\varphi\|_{L(X,\mathbb{R})} = \sup_{f \in X\setminus\{0\}} \frac{|\varphi(f)|}{\|f\|_X} \leq 2.$$

Since φ is linear, continuity follows from boundedness by Satz 2.2.1.

The sign function $f(x) = \frac{x}{|x|}$ is approximated pointwise by the sequence $(f_n)_{n\in\mathbb{N}}$ of functions $f_n \in X$ given by

$$f_n(t) = \begin{cases} -1, & \text{for } -1 \leq t < -\frac{1}{n}, \\ nt, & \text{for } -\frac{1}{n} \leq t < \frac{1}{n}, \\ 1, & \text{for } \frac{1}{n} \leq t \leq 1. \end{cases}$$

In particular, $\|f_n\|_X = 1$ for every $n \in \mathbb{N}$. Computing the integrals explicitly, or applying the dominated convergence theorem, we have

$$\lim_{n\to\infty} \varphi(f_n) = 2,$$

which proves that $\|\varphi\| = 2$.

(ii) Suppose there exists $f \in X$ with $\|f\|_X = 1$ and $|\varphi(f)| = 2$. Since φ is linear, we may assume $\varphi(f) = 2$, otherwise we replace f by $-f$. Then, the estimates

$$\left|\int_0^1 f(t)\,\mathrm{d}t\right| \leq \max_{x\in[-1,1]} |f(x)| = \|f\|_X = 1, \qquad \left|\int_{-1}^0 f(t)\,\mathrm{d}t\right| \leq 1,$$

imply by definition of φ that

$$\int_0^1 f(t)\,\mathrm{d}t = -\int_{-1}^0 f(t)\,\mathrm{d}t = 1. \tag{$*$}$$

Since f is bounded from above by 1 we can conclude from $(*)$ that $f|_{(0,1]} \equiv 1$. In fact, if $f(t^*) < 1$ for some $t^* \in (0,1]$, then $f < 1$ in some neighbourhood of t^* by continuity of f which together with the uniform bound $f \leq 1$ contradicts $(*)$.

Analogously, we conclude $f|_{[-1,0)} \equiv -1$ which (combined with $f|_{(0,1]} \equiv 1$) violates continuity of f at 0.

(iii) From (i) we know that $\varphi\colon X \to \mathbb{R}$ is a continuous linear functional. Therefore $N = \ker(\varphi)$ is a closed subspace of X.

By (ii) we know that there does not exist $f \in X$ with $\|f\|_X = 1$ and $|\varphi(f)| = 2$. Hence, by Problem 7.3, this proves that it does not exist any $y_0 \in N$ such that $\|x_0 - y_0\| = \operatorname{dist}(x_0, N)$.

Solution of 7.5: As a continuous linear operator $f\colon Y \subset H \to \mathbb{R}$ is closable. Let $\overline{f}$ be its closure. We claim that $D(\overline{f}) = \overline{Y}$. A priori, we only know $D(\overline{f}) \subset \overline{D(f)} = \overline{Y}$. Therefore, we consider $y \in \overline{Y}$ together with a sequence $(y_k)_{k\in\mathbb{N}}$ in Y such that $\|y - y_k\|_H \to 0$ as $k \to \infty$. From

$$|f(y_n) - f(y_m)| \leq \|f\|\|y_n - y_m\|_H,$$

we conclude that $(f(y_k))_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. Thus, there exists $z \in \mathbb{R}$ such that $f(y_k) \to z$ as $k \to \infty$. This means that (y, z) is in the closure of the graph of f and we conclude $y \in D(\overline{f})$. Moreover, by continuity of the norm,

$$|\overline{f}(y)| = \lim_{k\to\infty} |f(y_k)| \leq \lim_{k\to\infty} \|f\|\|y_k\|_H = \|f\|\|y\|_H \qquad \Longrightarrow \quad \|\overline{f}\| = \|f\|.$$

The argument above shows that in order to extend $f\colon Y \to \mathbb{R}$, we can first uniquely extend to $\overline{f}\colon \overline{Y} \to \mathbb{R}$ without changing the norm and then extend to $F\colon H \to \mathbb{R}$ with the advantage that we can now work with the *closed* subspace $\overline{Y}$. In fact, $(\overline{Y}, (\cdot,\cdot)_H)$ is a Hilbert space. This allows us to apply the Riesz representation theorem, namely there exists a unique $h \in \overline{Y}$ such that for all $y \in \overline{Y}$

$$\overline{f}(y) = (y, h)_H.$$

This suggests the extension

$$\begin{aligned} F\colon H &\to \mathbb{R} \\ x &\mapsto (x, h)_H \end{aligned}$$

which satisfies $\|F\| = \|h\|_H = \|\overline{f}\| = \|f\|$. Is this extension unique? If $\tilde{F}\colon H \to \mathbb{R}$ is another extension of $\overline{f}$ with $\|\tilde{F}\| = \|\overline{f}\|$ then it must be also of the form $\tilde{F}(x) = (x, \tilde{h})_H$ for some $\tilde{h} \in H$ by the Riesz representation theorem.

Since $\tilde{F}|_{\overline{Y}} = \overline{f} = F|_{\overline{Y}}$, we have

$$\forall y \in \overline{Y}: \quad 0 = F(y) - \tilde{F}(y) = (y, h)_H - (y, \tilde{h})_H = (y, h - \tilde{h})_H$$

which implies $h - \tilde{h} \in \overline{Y}^{\perp}$. Since $h \in \overline{Y}$ and $\|h\|_H = \|F\| = \|\tilde{F}\| = \|\tilde{h}\|$, we have

$$\|h\|_H^2 = \|\tilde{h}\|_H^2 = \|\tilde{h} - h + h\|_H^2 = \|\tilde{h} - h\|_H^2 + \|h\|_H^2,$$

where we used $(\tilde{h} - h) \perp h$. This implies $\|\tilde{h} - h\|_H^2 = 0$. Therefore, $\tilde{h} = h$ and F is the unique extension of f with $\|F\| = \|f\|$.

Solution of 7.6: Without loss of generality, we can assume $x = 0$. Otherwise we apply the translation $y \mapsto y - x$ which is an isometry in the entire space H.

(i) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in the convex set $Q \subset H$ with $\|x_n\| \to d = \operatorname{dist}(0, Q)$ as $n \to \infty$. By convexity of Q, the implications

$$x_n, x_m \in Q \qquad \Longrightarrow \quad \frac{x_n + x_m}{2} \in Q \qquad \Longrightarrow \quad \left\|\frac{x_n + x_m}{2}\right\|_H \geq \operatorname{dist}(0, Q) = d$$

hold for all $n, m \in \mathbb{N}$. Hence the parallelogram identity yields

$$\begin{aligned} \|x_n - x_m\|_H^2 &= 2\|x_n\|_H^2 + 2\|x_m\|_H^2 - \|x_n + x_m\|_H^2 \\ &= 2\|x_n\|_H^2 + 2\|x_m\|_H^2 - 4\left\|\frac{x_n + x_m}{2}\right\|_H^2 \leq 2\|x_n\|_H^2 + 2\|x_m\|_H^2 - 4d^2. \end{aligned}$$

From $2\|x_n\|_H^2 \to 2d^2$ as $n \to \infty$, we conclude that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in H.

(ii) Now we assume that the convex set $Q \subset H$ is closed. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in Q with $\|x_n\|_H \to d = \operatorname{dist}(0, Q)$ as $n \to \infty$. According to (i), it must be a Cauchy sequence. Since H is complete, $(x_n)_{n\in\mathbb{N}}$ converges to some $y \in H$. In fact, $y \in Q$ since Q is closed.

Suppose there is another point $\tilde{y} \in Q$ with $\|\tilde{y}\| = d$. Then, again by convexity and the parallelogram identity,

$$d^2 \leq \left\|\frac{y + \tilde{y}}{2}\right\|_H^2 \leq \left\|\frac{y + \tilde{y}}{2}\right\|_H^2 + \left\|\frac{y - \tilde{y}}{2}\right\|_H^2 = \frac{1}{2}\|y\|_H^2 + \frac{1}{2}\|\tilde{y}\|_H^2 = d^2$$

and we conclude that all the inequalities are in fact equalities, which implies

$$\left\|\frac{y - \tilde{y}}{2}\right\|_H^2 = 0.$$

Thus, $y = \tilde{y}$ and we have proven existence and uniqueness of $y \in Q$ with $\|y\|_H = d$.