8.1. Duality of sequence spaces. Consider the spaces

$$c_0 := \Big\{(x_k)_{k\in\mathbb{N}} \in \ell^\infty \;\Big|\; \lim_{k\to\infty} x_k = 0\Big\}, \qquad c := \Big\{(x_k)_{k\in\mathbb{N}} \in \ell^\infty \;\Big|\; \lim_{k\to\infty} x_k \text{ exists}\Big\}.$$

(i) Quick warm-up: Is $(c_0, \|\cdot\|_{\ell^\infty})$ a Banach space? Is $(c, \|\cdot\|_{\ell^\infty})$ a Banach space?

(ii) Show that the dual space of $(c_0, \|\cdot\|_{\ell^\infty})$ is *isometrically* isomorphic to $(\ell^1, \|\cdot\|_{\ell^1})$.

(iii) To which space is the dual space of $(c, \|\cdot\|_{\ell^\infty})$ isomorphic?

8.2. A result by Lions-Stampacchia. Let $(H, (\cdot,\cdot)_H)$ be a Hilbert space and let $\emptyset \neq K \subset H$ be a closed, convex subset. Let $f\colon H \to \mathbb{R}$ be a continuous linear functional and let $a\colon H \times H \to \mathbb{R}$ be a bilinear map satisfying

(i) $\forall x, y \in H: \quad a(x,y) = a(y,x)$

(ii) $\exists \Lambda > 0 \quad \forall x, y \in H: \quad |a(x,y)| \leq \Lambda \|x\|_H \|y\|_H$

(iii) $\exists \lambda > 0 \quad \forall x \in H: \quad a(x,x) \geq \lambda \|x\|_H^2.$

Consider the functional $J\colon H \to \mathbb{R}$ given by $J(x) = a(x,x) - 2f(x)$ and prove that there exists a unique $y_0 \in K$ such that the two following inequalities both hold:

$$\begin{aligned} \forall y \in K: &\quad J(y_0) \leq J(y), \\ \forall y \in K: &\quad a(y_0, y - y_0) \geq f(y - y_0). \end{aligned}$$

Moreover show that y_0 is equal to Px_0, where $x_0 \in H$ is such that $f(x) = a(x_0, x)$ and $P\colon H \to K$ is the operator mapping $x \in H$ to the unique point $Px \in K$ with $\|x - Px\|_H = \operatorname{dist}(x, K)$ (see Problem 7.6 (ii)).

8.3. Projection to convex sets. Let $(H, (\cdot,\cdot)_H)$ be a Hilbert space and let $\emptyset \neq K \subset H$ be a closed, convex subset. Let $P\colon H \to K$ be the operator which maps $x \in H$ to the unique point $Px \in K$ with $\|x - Px\|_H = \operatorname{dist}(x, K)$ (see Problem 7.6 (ii)).

(i) For every $x_1, x_2 \in H$ prove the inequality

$$\|Px_1 - Px_2\|_H \leq \|x_1 - x_2\|_H.$$

Hint. Use Problem 8.2.

(ii) Prove that

$$K = \bigcap_{x \in H} \{y \in H \mid (Px - x, y - Px)_H \geq 0\}.$$

8.4. Strict convexity.

Definition. A normed space $(X, \|\cdot\|_X)$ is called *strictly convex* if $\|\lambda x + (1-\lambda) y\|_X < 1$ holds for all $0 < \lambda < 1$ and all $x, y \in X$ with $x \neq y$ and $\|x\|_X = 1 = \|y\|_X$.

Let $(X, \|\cdot\|_X)$ be a normed space. The "abundance"-Lemma (Satz 4.2.1) states that

$$\forall x \in X \quad \exists x^* \in X^* : \quad \|x^*\|_{X^*}^2 = x^*(x) = \|x\|_X^2.$$

(i) Prove that if X^* (but not necessarily X) is strictly convex, then for all $x \in X$ there exists a *unique* $x^* \in X^*$ with $\|x^*\|_{X^*}^2 = x^*(x) = \|x\|_X^2$.

(ii) Find a counterexample for uniqueness of such x^*, if X^* is not strictly convex.

8.5. Uniform convexity ✍.

Definition. Let $(X, \|\cdot\|_X)$ be a Banach space and let $S = \{x \in X \mid \|x\|_X = 1\}$ be the unit sphere in X. The space $(X, \|\cdot\|_X)$ is called *uniformly convex* if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x, y \in S : \quad \|x - y\|_X > \varepsilon \implies \left\|\frac{x+y}{2}\right\|_X < 1 - \delta.$$

Remark. Uniform convexity is not to be confused with *strict convexity* defined in Problem 8.4.

(i) Prove that Hilbert spaces are uniformly convex.

(ii) Provide an example of a Banach space which is not uniformly convex.

8.6. Functional on the span of a sequence ✍. Let $(X, \|\cdot\|_X)$ be a normed space, let $(x_k)_{k\in\mathbb{N}}$ be a sequence in X and $(\alpha_k)_{k\in\mathbb{N}}$ a sequence in $\mathbb{R}$. Prove that the following statements are equivalent.

(i) There exists $\ell \in X^*$ satisfying $\ell(x_k) = \alpha_k$ for every $k \in \mathbb{N}$.

(ii) There exists $\gamma > 0$ such that for every sequence $(\beta_k)_{k\in\mathbb{N}}$ in $\mathbb{R}$ and every $n \in \mathbb{N}$ it holds

$$\left|\sum_{k=1}^{n} \beta_k \alpha_k\right| \leq \gamma \left\|\sum_{k=1}^{n} \beta_k x_k\right\|_X.$$

8. Solutions

Solution of 8.1:

(i) In Problem 2.4 (i) we have shown that the space $(c_0, \|\cdot\|_{\ell^\infty})$ is complete. To show completeness of $(c, \|\cdot\|_{\ell^\infty})$ it suffices to prove that c is closed in ℓ^∞. Let $x = (x_n)_{n\in\mathbb{N}} \in \overline{c}$. Then there exists a sequence $(x^{(k)})_{k\in\mathbb{N}}$ of sequences $x^{(k)} = (x_n^{(k)})_{n\in\mathbb{N}} \in c$ such that

$$\sup_{n\in\mathbb{N}} |x_n^{(k)} - x_n| = \|x^{(k)} - x\|_{\ell^\infty} \xrightarrow{k\to\infty} 0.$$

Given $\varepsilon > 0$, let $k_\varepsilon \in \mathbb{N}$ such that $\|x^{(k_\varepsilon)} - x\|_{\ell^\infty} < \varepsilon$. By definition, $x^{(k_\varepsilon)} \in c$ is a Cauchy sequence. Let $N_\varepsilon \in \mathbb{N}$ such that $|x_n^{(k_\varepsilon)} - x_m^{(k_\varepsilon)}| < \varepsilon$ for every $m, n \geq N_\varepsilon$. Then

$$|x_n - x_m| \leq |x_n - x_n^{(k_\varepsilon)}| + |x_n^{(k_\varepsilon)} - x_m^{(k_\varepsilon)}| + |x_m^{(k_\varepsilon)} - x_m| < 3\varepsilon$$

for every $m, n \geq N_\varepsilon$ which proves that x is a Cauchy sequence. Therefore, $x \in c$.

(ii) The linear map

$$\begin{aligned} \Psi\colon \ell^1 &\to c_0^* \\ y = (y_n)_{n\in\mathbb{N}} &\mapsto (\Psi y\colon c_0 \to \mathbb{R}) \\ &\qquad (x_n)_{n\in\mathbb{N}} \mapsto \sum_{n\in\mathbb{N}} x_n y_n \end{aligned}$$

is well-defined, since for every $y \in \ell^1$ and every $x = (x_n)_{n\in\mathbb{N}}$ in $(c_0, \|\cdot\|_{\ell^\infty})$ there holds

$$|(\Psi y)(x)| \leq \sum_{n\in\mathbb{N}} |x_n y_n| \leq \|x\|_{\ell^\infty} \|y\|_{\ell^1}.$$

This directly implies $\|\Psi y\|_{c_0^*} \leq \|y\|_{\ell^1}$. We claim $\|\Psi y\|_{c_0^*} = \|y\|_{\ell^1}$ for every $y \in \ell^1$. Indeed, given $y \in \ell^1$ we can apply Ψy to the sequence $x^{(k)} = (x_n^{(k)})_{n\in\mathbb{N}} \in c_0$ given by

$$x_n^{(k)} = \begin{cases} \frac{y_n}{|y_n|} & \text{if } n \leq k \text{ and } y_n \neq 0, \\ 0 & \text{else.} \end{cases}$$

and satisfying $\|x^{(k)}\|_{\ell^\infty} = 1$ to obtain

$$\lim_{k\to\infty} |\Psi(y)(x^{(k)})| = \lim_{k\to\infty} \sum_{n=1}^{k} |y_n| = \|y\|_{\ell^1} \qquad \Longrightarrow \|\Psi y\|_{c_0^*} = \sup_{\substack{x\in c_0 \\ \|x\|_{\ell^\infty}=1}} |\Psi(y)(x)| \geq \|y\|_{\ell^1}.$$

Therefore, Ψ is an isometry (it does not change the norm of any y) which also implies injectivity. To prove that Ψ is surjective, we show first, that every functional $f \in c_0^*$ is determined by its values on the elements $e^{(k)} = (e_n^{(k)})_{n\in\mathbb{N}} \in c_0$, where $e^{(k)} = (0, \ldots, 0, 1, 0, \ldots)$ has 1 at k-th position. Given $x = (x_n)_{n\in\mathbb{N}} \in c_0$, we have

$$\Big\| x - \sum_{k=1}^{N} x_k e^{(k)} \Big\|_{\ell^\infty} = \sup_{n>N} |x_n| \xrightarrow{N\to\infty} 0.$$

Continuity and linearity of f implies

$$f(x) = \lim_{N\to\infty} f\Big(\sum_{k=1}^{N} x_k e^{(k)}\Big) = \lim_{N\to\infty} \sum_{k=1}^{N} x_k f(e^{(k)}).$$

Given $f \in c_0^*$ we claim that $y := (f(e^{(k)}))_{k\in\mathbb{N}} \in \ell^1$ and $\Psi y = f$. Indeed, for any $N \in \mathbb{N}$

$$\sum_{k=1}^{N} \Big| f(e^{(k)}) \Big| = \sum_{k=1}^{\infty} x_k^{(N)} f(e^{(k)}) = f(x^{(N)}) \leq \|f\|_{c_0^*},$$

where $x_k^{(N)} = (x_k^{(N)})_{k\in\mathbb{N}} \in c_0$ with $\|x^{(N)}\|_{\ell^\infty} \leq 1$ is defined by

$$x_k^{(N)} = \begin{cases} \frac{f(e^{(k)})}{|f(e^{(k)})|} & \text{if } k \leq N \text{ and } f(e^{(k)}) \neq 0, \\ 0 & \text{else.} \end{cases}$$

Since N is arbitrary, we conclude $y \in \ell^1$. Moreover, given any $x = (x_k)_{k\in\mathbb{N}} \in c_0$ and y as above, we have

$$(\Psi y)(x) = \sum_{k\in\mathbb{N}} x_k y_k = \sum_{k\in\mathbb{N}} x_k f(e^{(k)}) = f(x)$$

which shows that Ψ is surjective and completes the prove that Ψ is a linear, bijective map, which is an isometry.

(iii) The dual space of $(c, \|\cdot\|_{\ell^\infty})$ is also isomorphic to $c_0^* \cong \ell^1$ but not isometrically. To construct an isomorphism $\Phi \colon c^* \to c_0^*$, we first consider the linear map

$$\begin{aligned} T \colon c &\to c_0 \\ x = (x_n)_{n\in\mathbb{N}} &\mapsto \Big(\lim_{n\to\infty} x_n, (x_1 - \lim_{n\to\infty} x_n), (x_2 - \lim_{n\to\infty} x_n), \ldots\Big). \end{aligned}$$

By definition of c and c_0, the map T is well-defined. T is continuous since

$$|\lim_{n\to\infty} x_n| \leq \|x\|_{\ell^\infty} \implies \|Tx\|_{\ell^\infty} \leq 2\|x\|_{\ell^\infty}.$$

Moreover, T is invertible with inverse

$$\begin{aligned} S \colon c_0 &\to c \\ y = (y_n)_{n\in\mathbb{N}} &\mapsto \Big((y_2 + y_1), (y_3 + y_1), (y_4 + y_1), \ldots\Big). \end{aligned}$$

Indeed, $STx = x$ is immediate and $TSy = y$ follows from $\lim_{n\to\infty}(y_n + y_1) = y_1$. Satisfying $\|Sy\|_{\ell^\infty} \leq 2\|y\|_{\ell^\infty}$, the map S is also continuous. We are ready to define

$$\begin{aligned} \Phi \colon c^* &\to c_0^* \\ f &\mapsto f \circ S. \end{aligned}$$

As composition of linear maps, Φ is linear. It is also continuous since

$$\begin{aligned} |(\Phi f)(y)| = |f(Sy)| \leq \|f\|_{c^*} \|Sy\|_{\ell^\infty} \leq 2\|f\|_{c^*} \|y\|_{\ell^\infty} \quad &\implies \quad \|\Phi f\|_{\ell^\infty} \leq 2\|f\|_{c^*} \\ &\implies \quad \|\Phi\|_{L(c^*, c_0^*)} \leq 2. \end{aligned}$$

Finally, by the construction above, Φ bijective with inverse $\Phi^{-1} \colon g \mapsto g \circ T$.

Solution of 8.2:

Claim 1. Given $f \in H^*$, there exists a unique $x_0 \in H$ such that for all $x \in H$

$$J(x) := a(x,x) - 2f(x) = a(x-x_0, x-x_0) - a(x_0,x_0).$$

Proof. Since a is bilinear and satisfies (ii) and (iii) the Lax-Milgram Theorem applies. ((ii) implies continuity of a). In particular, since $f \in H^*$, there exists a unique $x_0 \in H$ satisfying $a(x_0,x) = f(x)$ for all $x \in H$ by Korollar 4.3.1. (The same follows from claim 2 below and the Riesz representation theorem applied in $(H, a(\cdot,\cdot))$). Moreover,

$$\begin{aligned} J(x) = a(x,x) - 2f(x) &= a(x,x) - 2a(x_0,x) \\ &= a(x-x_0,x) - a(x_0,x) \\ &= a(x-x_0,x-x_0) + a(x-x_0,x_0) - a(x,x_0) \\ &= a(x-x_0,x-x_0) - a(x_0,x_0). \end{aligned}$$ ☐

Claim 2. $(H, a(\cdot,\cdot))$ is a Hilbert space.

Proof. By assumption (i) the bilinear map a is symmetric. By (ii) and (iii), we have

$$\lambda \|x\|_H^2 \leq a(x,x) \leq \Lambda \|x\|_H^2 \tag{$*$}$$

which shows $a(x,x) \geq 0$ and $a(x,x) = 0 \Leftrightarrow x = 0$. Therefore, $a(\cdot,\cdot)$ is a scalar product on H. In fact, $(*)$ implies that the induced norm $\|\cdot\|_a = \sqrt{a(\cdot,\cdot)}$ is *equivalent* to $\|\cdot\|_H$, where we recall the definition of equivalent norms from Problem 1.1. It is easy to check that equivalent norms have the same Cauchy sequences and induce the same notion of convergence. Therefore, $(H, \|\cdot\|_a)$ is complete since $(H, \|\cdot\|_H)$ is complete and the claim follows. ☐

By assumption, the set $K \subset H$ is convex and closed in $(H, \|\cdot\|_H)$. Since the two norms are equivalent, K is also closed in $(H, \|\cdot\|_a)$ and we can apply the result of Problem 7.6 (ii) in the Hilbert space $(H, a(\cdot,\cdot))$ with the point x_0 from Claim 1. Namely, there exists a unique $y_0 \in K$ satisfying

$$\|x_0 - y_0\|_a = \inf_{y \in K} \|x_0 - y\|_a \tag{$\dagger$}$$

By Claim 1 we have for arbitrary $y \in K$

$$J(y_0) = \|y_0 - x_0\|_a^2 - \|x_0\|_a^2 \leq \|y - x_0\|_a^2 - \|x_0\|_a^2 = J(y)$$

which proves the first inequality.

The second inequality claims non-negativity of

$$\begin{aligned} a(y_0, y - y_0) - f(y - y_0) &= a(y_0, y - y_0) - a(x_0, y - y_0) \\ &= a(y_0 - x_0, y - y_0) \end{aligned} \tag{$\ddagger$}$$

for every $y \in K$. Since $y_0 \in K$ we have $ty + (1-t)y_0 \in K$ for every fixed $y \in K$ and every $t \in [0,1]$ by convexity of K. We consider the map $g\colon [0,1] \to \mathbb{R}$ given by

$$g(t) = \Big\| x_0 - \Big(ty + (1-t)y_0\Big)\Big\|_a^2 = \Big\| x_0 - y_0 + t(y_0 - y)\Big\|_a^2.$$

By definition (†) of y_0, and since $ty + (1-t)y_0 \in K$ by convexity, g has a minimum at the boundary point $t = 0$ which implies $g'(0) \geq 0$. We compute

$$\begin{aligned} g'(t) &= 2a\Big(x_0 - y_0 + t(y_0 - y), y_0 - y\Big), \\ 0 \leq g'(0) &= 2a(x_0 - y_0, y_0 - y) = 2a(y_0 - x_0, y - y_0). \end{aligned}$$

Since $y \in K$ is arbitrary, the second inequality follows.

Solution of 8.3:

(i) Given any $x_0 \in H$ and defining $a(\cdot,\cdot) = (\cdot,\cdot)_H$ and $f(\cdot) = (x_0, \cdot)$. The second inequality (‡) proved in Problem 8.2, with $y_0 = Px_0$, gives

$$\forall y \in K: \quad (Px_0 - x_0, y - Px_0)_H \geq 0. \tag{$\sharp$}$$

Now, given $x_1, x_2 \in H$, we apply inequality (♯) twice, first with $x_0 = x_1 \in H$ and $y = Px_2 \in K$ and then with $x_0 = x_2 \in H$ and $y = Px_1 \in K$ to obtain

$$\begin{aligned} 0 &\leq (Px_1 - x_1, Px_2 - Px_1)_H + (Px_2 - x_2, Px_1 - Px_2)_H \\ &= (Px_2 - Px_1 + x_1 - x_2, Px_1 - Px_2)_H \\ &= (x_1 - x_2, Px_1 - Px_2)_H - \|Px_1 - Px_2\|_H^2, \\ \implies \|Px_1 - Px_2\|_H^2 &\leq (x_1 - x_2, Px_1 - Px_2)_H \\ &\leq \|x_1 - x_2\|_H \|Px_1 - Px_2\|_H, \\ \implies \|Px_1 - Px_2\|_H &\leq \|x_1 - x_2\|_H. \end{aligned}$$

(ii) Let us prove the two inclusions separately.

"$\subseteq$" Let $y \in K$. Then $(Px - x, y - Px)_H \geq 0$ for any $x \in H$ by (♯).

"$\supseteq$" Let $y \in H \setminus K$. Then, choosing $x = y$, we have $Py \neq y$ which implies

$$(Py - y, y - Py)_H = -\|Py - y\|_H^2 < 0$$

and shows that y is not element of the intersection on the right hand side.

Solution of 8.4:

(i) Given $0 \neq x \in X$, let $x^* \in X^*$ and $y^* \in X^*$ satisfy

$$\|x^*\|_{X^*}^2 = x^*(x) = \|x\|_X^2 = y^*(x) = \|y^*\|_{X^*}^2.$$

Then

$$\|x\|_X^2 = \frac{1}{2}\Big(x^*(x) + y^*(x)\Big) = \Big(\tfrac{1}{2}x^* + \tfrac{1}{2}y^*\Big)(x) \leq \Big\|\tfrac{1}{2}x^* + \tfrac{1}{2}y^*\Big\|_{X^*}\|x\|_X.$$

We divide by $\|x\|_X^2$ to obtain

$$1 \leq \left\|\frac{x^*}{2\|x\|_X} + \frac{y^*}{2\|x\|_X}\right\|_{X^*}, \qquad \left\|\frac{x^*}{\|x\|_X}\right\|_{X^*} = \frac{\|x^*\|_{X^*}}{\|x\|_X} = 1 = \left\|\frac{y^*}{\|x\|_X}\right\|_{X^*}.$$

If $x^* \neq y^*$ then $\lambda = \frac{1}{2}$ in the definition of strict convexity of X^* yields the contradiction

$$\left\|\frac{x^*}{2\|x\|_X} + \frac{y^*}{2\|x\|_X}\right\|_{X^*} < 1.$$

(ii) Consider the space $(\mathbb{R}^2, \|\cdot\|_\infty)$, where we define $\|p\|_\infty := \max\{|p_1|, |p_2|\}$ for every $p = (p_1, p_2) \in \mathbb{R}^2$. Let $x = (1, 1)$. Then, $\|x\|_\infty = 1$ and the functionals

$$x^* \colon \mathbb{R}^2 \to \mathbb{R}^2, \quad (p_1, p_2) \mapsto p_1 \qquad\qquad y^* \colon \mathbb{R}^2 \to \mathbb{R}^2, \quad (p_1, p_2) \mapsto p_2$$

both satisfy $x^*(x) = y^*(x) = 1 = \|x\|_\infty^2$ and

$$\|x^*\|_{X^*} = \sup_{\|p\|_\infty \leq 1} |x^*(p)| = \sup_{|p_1|, |p_2| \leq 1} |p_1| = 1 = \sup_{|p_1|, |p_2| \leq 1} |p_2| = \|y^*\|_{X^*}.$$

Solution of 8.5:

(i) Let $(H, (\cdot, \cdot))$ be a Hilbert space. Let $\varepsilon > 0$. For all $x, y \in H$ with $\|x\| = 1 = \|y\|$ and $\|x - y\| > \varepsilon$, the parallelogram identity (see Problem 1.2) implies

$$\left\|\frac{x+y}{2}\right\|^2 = 2\left\|\frac{x}{2}\right\|^2 + 2\left\|\frac{y}{2}\right\|^2 - \left\|\frac{x-y}{2}\right\|^2 < \frac{1}{2} + \frac{1}{2} - \frac{\varepsilon^2}{4}$$

$$\implies \left\|\frac{x+y}{2}\right\| \leq \left(1 - \frac{\varepsilon^2}{4}\right)^{\frac{1}{2}}.$$

(ii) We claim that the Banach space $(L^\infty(\mathbb{R}), \|\cdot\|_{L^\infty})$ is not uniformly convex. Consider the characteristic functions $u = \chi_{[0,1]}$ and $v = \chi_{[t,1+t]}$ and $\varepsilon = \frac{1}{2}$. For any $0 < t < 1$, one has $\|u\|_{L^\infty} = 1 = \|v\|_{L^\infty}$ and $\|u - v\|_{L^\infty} = 1 > \varepsilon$, but $\|\frac{1}{2}(u + v)\|_{L^\infty} = 1$.

In analogy to the counterexample in Problem 8.4 (ii), the finite dimensional Banach space $(\mathbb{R}^2, \|\cdot\|_\infty)$, where we define $\|p\|_\infty := \max\{|p_1|, |p_2|\}$ for every $p = (p_1, p_2) \in \mathbb{R}^2$, is not uniformly convex. Indeed, given $x = (1, 1)$ and $y = (1, 0)$, we have $\|x\|_\infty = 1 = \|y\|_\infty$ and $\|x - y\|_\infty = 1$ but $\|\frac{1}{2}(x + y)\|_\infty = \|(1, \frac{1}{2})\|_\infty = 1$.

Solution of 8.6: Let $(x_k)_{k\in\mathbb{N}}$ be a sequence in $(X,\|\cdot\|_X)$ and $(\alpha_k)_{k\in\mathbb{N}}$ a sequence in $\mathbb{R}$.

"(i) ⇒ (ii)" Let $(\beta_k)_{k\in\mathbb{N}}$ be any sequence in $\mathbb{R}$. Given $\ell\in X^*$ with $\ell(x_k)=\alpha_k$ for every $k\in\mathbb{N}$, we can make the following estimate for any $n\in\mathbb{N}$.

$$\Big|\sum_{k=1}^n \beta_k\alpha_k\Big| = \Big|\ell\Big(\sum_{k=1}^n \beta_k x_k\Big)\Big| \leq \|\ell\|_{X^*}\Big\|\sum_{k=1}^n \beta_k x_k\Big\|_X$$

and statement (ii) follows with $\gamma=\|\ell\|_{X^*}$.

"(ii) ⇒ (i)" Every element of the subspace $U=\operatorname{span}\{x_k \mid k\in\mathbb{N}\}\subset X$ is of the form

$$y=\sum_{k=1}^n \beta_k x_k$$

(for not necessarily unique $n\in\mathbb{N}$ and $\beta_k\in\mathbb{R}$). However, assumption (ii) implies that

$$\begin{aligned}\tilde{\ell}\colon U &\to \mathbb{R}\\ y=\sum_{k=1}^n \beta_k x_k &\mapsto \sum_{k=1}^n \beta_k\alpha_k\end{aligned}$$

is well-defined. In fact, if

$$\sum_{k=1}^n \beta_k x_k = \sum_{k=1}^m \beta'_k x_k,$$

then, setting $N:=\max\{n,m\}$ and $\beta_k=0$ for $k>n$ respectively $\beta'_k=0$ for $k>m$,

$$\begin{aligned}\Big|\sum_{k=1}^n \beta_k\alpha_k - \sum_{k=1}^m \beta'_k\alpha_k\Big| &= \Big|\sum_{k=1}^N (\beta_k-\beta'_k)\alpha_k\Big|\\ &\leq \gamma\Big\|\sum_{k=1}^N (\beta_k-\beta'_k)x_k\Big\|_X = \gamma\Big\|\sum_{k=1}^n \beta_k x_k - \sum_{k=1}^m \beta'_k x_k\Big\| = 0.\end{aligned}$$

Moreover, assumption (ii) implies that the linear functional $\tilde{\ell}$ is continuous on $(U,\|\cdot\|_X)$ with $\|\tilde{\ell}\|_{U^*}\leq\gamma$. The Hahn-Banach Theorem implies (Satz 4.1.3) that there exists an extension $\ell\in X^*$ with $\|\ell\|_{X^*}=\|\tilde{\ell}\|_{U^*}\leq\gamma$ and $\ell(x_k)=\tilde{\ell}(x_k)=\alpha_k$ for every $k\in\mathbb{N}$.