9.1. Representation of a convex set. Let $(X, \|\cdot\|_X)$ be a normed space and let $\emptyset \neq Q \subset X$ be an open, convex subset containing the origin. Prove that there exists a subset $\Upsilon \subset X^*$ such that

$$Q = \bigcap_{f \in \Upsilon} \{x \in X \mid f(x) < 1\},$$

which means that Q is an intersection of open, affine half-spaces.

9.2. Extremal subsets.

Definition. Let X be a vector space and $K \subset X$ any subset. A subset $M \subset K$ is called *extremal subset* of K if

$$\forall x_1, x_0 \in K \quad \forall \lambda \in (0,1): \quad \Bigl(\lambda x_1 + (1-\lambda)x_0 \in M \;\Rightarrow\; x_1, x_0 \in M\Bigr)$$

If M consists of only one point $M = \{y\}$, we say that y is an *extremal point* of K.

Let X be vector space and let $K \subset X$ be a convex subset with more than one element.

(i) Assume $K \subset \mathbb{R}^2$ is also closed. Prove that the set E of all extremal points of K is closed.

(ii) Is the statement of (i) also true in $\mathbb{R}^3$?

(iii) Given an extremal subset $M \subset K$ of K, prove that $K \setminus M$ is convex.

(iv) Prove that $y \in K$ is an extremal point of K if and only if $K \setminus \{y\}$ is convex.

(v) If $N \subset K$ and $K \setminus N$ are both convex, does it follow that N is extremal?

9.3. Weak sequential continuity of linear operators. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces and let $T \colon X \to Y$ be a linear operator. Prove that the following statements are equivalent.

(i) T is continuous.

(ii) For every sequence $(x_n)_{n\in\mathbb{N}}$ in X, weak convergence $x_n \xrightarrow{w} x$ in X for $n \to \infty$ implies weak convergence $Tx_n \xrightarrow{w} Tx$ in Y for $n \to \infty$.

9.4. Weak convergence in finite dimensions. Let $(X, \|\cdot\|_X)$ be a normed space of *finite* dimension $\dim X = d < \infty$. Let $x \in X$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Prove that weak convergence $x_n \xrightarrow{w} x$ for $n \to \infty$ implies $\|x_n - x\|_X \to 0$ for $n \to \infty$.

9.5. Weak convergence in Hilbert spaces. Let $(H, (\cdot,\cdot)_H)$ be a real, infinite dimensional Hilbert space. Let $x \in H$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in H.

(i) Prove that weak convergence $x_n \xrightarrow{w} x$ in H and convergence of the norms $\|x_n\|_H \to \|x\|_H$ in $\mathbb{R}$ implies (strong) convergence $x_n \to x$ in H, i. e. $\|x_n - x\|_H \to 0$.

(ii) Suppose $x_n \xrightarrow{w} x$ and $\|y_n - y\|_H \to 0$, where $(y_n)_{n\in\mathbb{N}}$ is another sequence in H and $y \in H$. Prove that $(x_n, y_n)_H \to (x, y)_H$.

(iii) Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal system of $(H, (\cdot,\cdot)_H)$. Prove $e_n \xrightarrow{w} 0$ as $n \to \infty$.

(iv) Given any $x \in H$ with $\|x\|_H \leq 1$, prove that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in H satisfying $\|x_n\|_H = 1$ for all $n \in \mathbb{N}$ and $x_n \xrightarrow{w} x$ as $n \to \infty$.

(v) Let the functions $f_n\colon [0, 2\pi] \to \mathbb{R}$ be given by $f_n(t) = \sin(nt)$ for $n \in \mathbb{N}$. Prove that $f_n \xrightarrow{w} 0$ in $L^2([0, 2\pi])$ as $n \to \infty$.

9.6. Sequential closure ✍. Let X be a set and τ a topology on X. Given a subset $\Omega \subset X$, we use the notation

$$\overline{\Omega}_\tau := \bigcap_{\substack{A\supset\Omega,\\ X\setminus A\in\tau}} A$$

for the closure of Ω in the topology τ and

$$\overline{\Omega}_{\tau\text{-seq}} := \{x \in X \mid \exists (x_n)_{n\in\mathbb{N}} \text{ in } \Omega: \ x_n \xrightarrow{\tau} x \text{ as } n \to \infty\}$$

for the sequential closure of Ω induced by the topology τ, which is based on the notion of convergence in topological spaces:

$$(x_n \xrightarrow{\tau} x) \quad\Leftrightarrow\quad (\forall U \in \tau,\ x \in U \quad \exists N \in \mathbb{N} \quad \forall n \geq N: \quad x_n \in U).$$

(i) Prove that if $A \subset X$ is closed, then A is sequentially closed. Deduce the inclusion $\overline{\Omega}_{\tau\text{-seq}} \subset \overline{\Omega}_\tau$ for any subset $\Omega \subset X$.

(ii) Let $(X, \tau) = (\ell^2, \tau_{\mathrm{w}})$, where τ_{w} denotes the weak topology on ℓ^2. Find a set $\Omega \subset \ell^2$ for which the inclusion $\overline{\Omega}_{\text{w-seq}} \subset \overline{\Omega}_{\mathrm{w}}$ proven in (i) is strict. 💎

9.7. Convex hull ⚙.

Definition. Let $(X, \|\cdot\|_X)$ be a normed space. The *convex hull* of $A \subset X$ is defined as

$$\operatorname{conv}(A) := \bigcap_{\substack{B\supset A,\\ B \text{ convex}}} B$$

Recall the following representation theorem for convex hulls

$$\operatorname{conv}(A) = \Big\{\sum_{k=1}^{n} \lambda_k x_k \;\Big|\; n \in \mathbb{N},\ x_1, \ldots, x_n \in A,\ \lambda_1, \ldots, \lambda_n \geq 0,\ \sum_{k=1}^{n} \lambda_k = 1\Big\}.$$

(i) Using the representation of the convex hull above, prove Mazur's Lemma: If $(x_k)_{k\in\mathbb{N}}$ is a sequence in X satisfying $x_k \xrightarrow{w} x$ as $k \to \infty$, then there exists a sequence $(y_n)_{n\in\mathbb{N}}$ of convex linear combinations

$$y_n = \sum_{k=1}^{c(n)} a_{kn} x_k, \quad c(n) \in \mathbb{N}, \quad a_{kn} \geq 0 \text{ for } k = 1, \ldots, c(n), \quad \sum_{k=1}^{c(n)} a_{kn} = 1,$$

such that $\|y_n - x\|_X \to 0$ as $n \to \infty$.

(ii) Let $(X, \|\cdot\|_X)$ be a normed space and let $A, B \subset X$ be compact, convex subsets. Using the representation of the convex hull above, prove that $\operatorname{conv}(A \cup B)$ is compact.

9. Solutions

Solution of 9.1: Given the normed space $(X, \|\cdot\|_X)$, the non-trivial, open, convex subset $Q \subset X$ and the Minkowski functional

$$\begin{aligned} p\colon X &\to \mathbb{R} \\ x &\mapsto \inf\{\lambda > 0 \mid \tfrac{1}{\lambda}x \in Q\}, \end{aligned}$$

we define the set

$$\Upsilon := \{f \in X^* \mid \forall x \in X: \ f(x) \le p(x)\}$$

and claim that

$$Q = \bigcap_{f \in \Upsilon} \{x \in X \mid f(x) < 1\}.$$

"⊆" Let $x \in Q$. Since Q is open, we have $p(x) < 1$. For every $f \in \Upsilon$ we have $f(x) \le p(x)$ by definition. This proves $f(x) < 1$ for every $f \in \Upsilon$.

"⊇" Suppose $x_0 \notin Q$. We hope to find some $f \in \Upsilon$ with $f(x_0) \ge 1$. Towards that end, we define the functional

$$\begin{aligned} \ell\colon \operatorname{span}(\{x_0\}) &\to \mathbb{R} \\ tx_0 &\mapsto t. \end{aligned}$$

Since Q is convex and contains the origin, we have $p(x_0) \ge 1$. In particular, we have

$$\begin{aligned} \forall t \ge 0: &\quad \ell(tx_0) = t \le t\,p(x_0) = p(tx_0), \\ \forall t < 0: &\quad \ell(tx_0) = t < 0 \le p(tx_0). \end{aligned}$$

The Hahn-Banach theorem implies that there exists a linear functional $f\colon X \to \mathbb{R}$ which agrees with ℓ on $\operatorname{span}(\{x_0\})$ and satisfies $f(x) \le p(x)$ for every $x \in X$. Is f continuous? Since Q is open and contains the origin, there exists $r > 0$ such that $B_r(0) \subset Q$. Thus, $\frac{1}{\lambda}x \in Q$ with $\lambda = \frac{2}{r}\|x\|_X$ and the definition of p implies that

$$f(x) \le p(x) \le \tfrac{2}{r}\|x\|_X$$

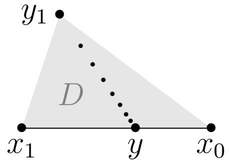
which yields that f is continuous and therefore $f \in \Upsilon$. Since $f(x_0) = 1$, the claim follows.

Solution of 9.2:

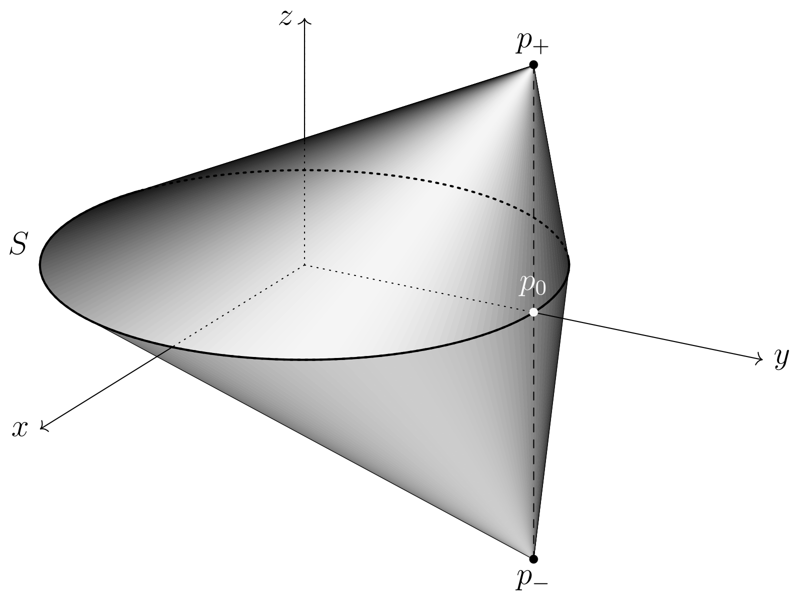
(i) It is clear that the set E of extremal points of the closed, convex subset $K \subset \mathbb{R}^2$ must be a subset of the boundary ∂K of K because the center of every ball contained in K is a convex combination of other points in this ball.

Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in E which converges to some $y \in K$. Suppose $y \notin E$. Then there exist distinct points $x_1, x_0 \in K$ and some $0 < \lambda < 1$ such that $\lambda x_1 + (1-\lambda)x_0 = y$. For any $n \in \mathbb{N}$, the point y_n is extremal and therefore cannot lie on the segment between

x_1 and x_0. Intuitively, the sequence $(y_n)_{n\in\mathbb{N}}$ must approach y from "above" or "below" this segment. By restriction to a subsequence, we can assume that all y_n strictly lie on the same side of the affine line through x_1 and x_2. By convexity of K, the triangle $D = \operatorname{conv}\{x_1, x_0, y_1\}$ is a subset of K. The arguments above and the convergence $y_n \to y$ imply that for $n \in \mathbb{N}$ sufficiently large, y_n is in the interior of D and thus in the interior of K. This however contradicts $y_n \in E \subset \partial K$. We conclude $y \in E$ which proves that E is closed.



(ii) The set of extremal points of a closed, convex subset in $\mathbb{R}^3$ is not necessarily closed: Let $S = \{(x, y, 0) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$ and $p_\pm = (0, 1, \pm 1)$. The set of extremal points of $\operatorname{conv}(S \cup \{p_+, p_-\})$ is $E = \{p_+, p_-\} \cup S \setminus p_0$, where $p_0 = (0, 1, 0) = \frac{1}{2}p_+ + \frac{1}{2}p_-$.



(iii) Let $K \subset X$ be convex and $M \subset K$ an extremal subset of K. Suppose, $K \setminus M$ is not convex. Then there are points $x_1, x_0 \in K \setminus M$ such that $x := \lambda x_1 + (1-\lambda)x_0 \notin K \setminus M$ for some $0 < \lambda < 1$. Since K is convex, $x \in K$ and hence $x \in M$. However, this contradicts $x_1, x_0 \notin M$ by definition of extremal subset.

(iv) If $y \in K$ is an extremal point of K, then $\{y\} \subset K$ is an extremal subset of K and (iii) implies that $K \setminus \{y\}$ is convex. Conversely, if $y \in K$ is not an extremal point of K, then by definition there exist $x_1, x_0 \in K \setminus \{y\}$ and some $0 < \lambda < 1$ such that $y = \lambda x_1 + (1-\lambda)x_0$ which shows that $K \setminus \{y\}$ is not convex.

(v) No, the interval $K = [-1, 1] \subset \mathbb{R}$, the subset $N = [-1, 0] \subset K$ and the difference $K \setminus N = (0, 1]$ are all convex but N is not an extremal subset of K since $\frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0 \in N$ but $1 \notin N$.

Solution of 9.3:

"(i) ⇒ (ii)" Let $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ be normed spaces. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X such that $x_n \xrightarrow{\text{w}} x$ for some $x\in X$. Let $f\in Y^*$ be arbitrary. If $T\colon X\to Y$ is a continuous linear operator, then $f\circ T\in X^*$ and weak convergence of $(x_n)_{n\in\mathbb{N}}$ implies

$$\lim_{n\to\infty} f(Tx_n) = \lim_{n\to\infty}(f\circ T)(x_n) = (f\circ T)(x) = f(Tx),$$

which proves weak convergence of $(Tx_n)_{n\in\mathbb{N}}$ in Y.

"(ii) ⇒ (i)" If the linear operator $T\colon X\to Y$ is not continuous, then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $\|x_n\|_X\leq 1$ and $\|Tx_n\|_Y\geq n^2$ for every $n\in\mathbb{N}$. Then $\frac{1}{n}x_n\to 0$ in X (in particular weakly) but $(T(\frac{1}{n}x_n))_{n\in\mathbb{N}}$ is unbounded in Y and therefore cannot be weakly convergent (Satz 4.6.1.).

Solution of 9.4: Let $e_1,\ldots,e_d$ be a basis for the finite dimensional normed space $(X,\|\cdot\|_X)$. Then, every element $x\in X$ is of the form $x=\sum_{k=1}^d x^k e_k$ for uniquely determined $x^1,\ldots,x^d\in\mathbb{R}$ (the superscripts are upper indices, not exponents). For $k\in\{1,\ldots,d\}$ we consider the linear maps $e_k^*\colon X\to\mathbb{R}$ given by $e_k^*(x)=x^k$. In fact, $e_k^*\in X^*$ since $|e_k^*(x)|=|x^k|\leq\|x\|_1$, where $\|x\|_1:=\sum_{k=1}^d|x^k|$ defines a norm on X which must be equivalent to $\|\cdot\|_X$ since X is finite dimensional.

If $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that $x_n\xrightarrow{\text{w}} x$ for some $x\in X$ as $n\to\infty$, then

$$\forall k\in\{1,\ldots,d\}: \quad \lim_{n\to\infty} x_n^k = \lim_{n\to\infty} e_k^*(x_n) = e_k^*(x) = x^k.$$

This implies $\|x_n-x\|_1\to 0$ and by equivalence of norms $\|x_n-x\|_X\to 0$ as $n\to\infty$.

Solution of 9.5:

(i) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in the Hilbert space $(H,(\cdot,\cdot)_H)$ such that $x_n\xrightarrow{\text{w}} x$ for some $x\in H$ and such that $\|x_n\|_H\to\|x\|_H$ as $n\to\infty$. Since $(x,\cdot)_H\in H^*$, weak convergence implies $(x,x_n)_H\to(x,x)_H=\|x\|_H^2$ as $n\to\infty$ and we have

$$\|x_n-x\|_H^2 = (x_n-x,x_n-x)_H = \|x_n\|_H^2 - 2(x,x_n)_H + \|x\|_H^2 \xrightarrow{n\to\infty} 0.$$

(ii) Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be sequences in H and $x,y\in H$ such that $x_n\xrightarrow{\text{w}} x$ and $\|y_n-y\|_H\to 0$ as $n\to\infty$. Weak convergence $x_n\xrightarrow{\text{w}} x$ implies in particular, that $(x_n,y)_H\to(x,y)_H$ as $n\to\infty$ and that there exists a finite constant C such that $\|x_n\|_H\leq C$ for all $n\in\mathbb{N}$. Thus,

$$\begin{aligned}\Big|(x_n,y_n)_H-(x,y)_H\Big| &= \Big|(x_n,y_n-y)_H+(x_n,y)_H-(x,y)_H\Big|\\ &\leq C\|y_n-y\|_H + \Big|(x_n,y)_H-(x,y)_H\Big| \xrightarrow{n\to\infty} 0.\end{aligned}$$

(iii) Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal system of the infinite dimensional Hilbert space $(H,(\cdot,\cdot)_H)$. Then, Bessel's inequality

$$\sum_{n=0}^{\infty}|(x,e_n)_H|^2 \leq \|x\|_H^2$$

implies $(x,e_n)_H \to 0$ as $n\to\infty$ for any $x\in H$. Since by the Riesz representation theorem any $f\in H^*$ satisfies $f(e_n)=(x,e_n)_H$ for a unique $x\in H$, we obtain $e_n \xrightarrow{\text{w}} 0$.

(iv) Let $x\in H$ satisfy $\|x\|_H\leq 1$. If $x=0$, then any orthonormal system converges weakly to x by (iii). If $x\neq 0$, then an orthonormal system $(e_n)_{n\in\mathbb{N}}$ of H with $e_1=\|x\|_H^{-1}x$ can be constructed via the Gram-Schmidt algorithm. For $n\in\mathbb{N}$, let

$$x_n := x + \Big(\sqrt{1-\|x\|_H^2}\Big)e_{n+1}$$

Then, since $x\perp e_{n+1}$, we have $\|x_n\|^2=\|x\|_H^2+(1-\|x\|_H^2)=1$ for every $n\in\mathbb{N}$. Moreover, $x_n \xrightarrow{\text{w}} x$ follows from $e_{n+1}\xrightarrow{\text{w}} 0$ as $n\to\infty$ by (iii).

(v) Given $f_n\colon[0,2\pi]\to\mathbb{R}$ as in the statement, $(\sqrt{\frac{1}{\pi}}f_n)_{n\in\mathbb{N}}$ is an orthonormal system of $L^2([0,2\pi])$, because

$$\begin{aligned}\int_0^{2\pi}\sin(mt)\sin(nt)\,\mathrm{d}t &= \frac{1}{2}\int_0^{2\pi}\cos\big((m-n)t\big)-\cos\big((m+n)t\big)\,\mathrm{d}t\\ &= \begin{cases}0, & \text{if } m\neq n,\\ \pi, & \text{if } m=n\end{cases}\end{aligned}$$

holds for any $m,n\in\mathbb{N}$. By (iii) weak convergence $f_n\xrightarrow{\text{w}}0$ as $n\to\infty$ follows.

Solution of 9.6:

(i) Let (X,τ) be a topological space and let $A\subset X$ be closed. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in A such that $x_n\xrightarrow{\tau}x$ as $n\to\infty$ for some $x\in X$. Suppose $x\notin A$. Then, $U:=X\setminus A$ is an open set in τ containing x. Convergence $x_n\xrightarrow{\tau}x$ implies that there exists $N\in\mathbb{N}$ such that $x_N\in U$. This however contradicts $x_N\in A$. Thus, $x\in A$ and we have proven that A is sequentially closed.

The set $\overline{\Omega}_\tau\subset X$ is closed, hence it is sequentially closed thanks to the first part of the exercise. Moreover $\overline{\Omega}_\tau$ contains Ω and thus $\overline{\Omega}_{\tau\text{-seq}}\subset\overline{\Omega}_\tau$, since every $x\in\overline{\Omega}_{\tau\text{-seq}}$ is the limit of a sequence $(x_n)_{n\in\mathbb{N}}\subset\Omega$ and $\overline{\Omega}_\tau$ is sequentially closed.

(ii) We construct a set $\Omega\subset\ell^2$ such that $(0)\in\overline{\Omega}_{\text{w}}$ but no sequences in Ω converge weakly to zero, i.e. $(0)\notin\overline{\Omega}_{\text{w-seq}}$. (Here we denote $(0):=(0,0,\dots)\in\ell^2$.)

For $n\in\mathbb{N}$ and $2\leq m\in\mathbb{N}$, let $x^{(n,m)}=(\frac{1}{n},0,\dots,0,n,0,\dots)\in\ell^2$, where the entry "$n$" is at m-th position. By the Riesz representation theorem, any $f\in(\ell^2)^*$ is of the form $f=(\cdot,y)_{\ell^2}$ for some $y\in\ell^2$. For any $y\in\ell^2$ and any $2\leq m,n\in\mathbb{N}$, we have

$$(x^{(n,m)},y)_{\ell^2}=\tfrac{1}{n}y_1+ny_m. \tag{$*$}$$

Define $\Omega = \{x^{(n,m)} \mid n,m \in \mathbb{N},\ m \geq 2\}$. Let $(x^{(n_k,m_k)})_{k\in\mathbb{N}}$ be any (fixed) sequence in Ω. Towards a contradiction, suppose $x^{(n_k,m_k)} \xrightarrow{\mathrm{w}} (0)$ as $k \to \infty$. From $(*)$ we conclude $n_k \to \infty$ and $m_k \to \infty$ as $k \to \infty$. (Note that for $y \in \ell^2$ we have $y_m \to 0$ as $m \to \infty$.) But then $\|x^{(n_k,m_k)}\|_{\ell^2}^2 = n_k^{-2} + n_k^2 \to \infty$ as $k \to \infty$ and we derived a contradiction to the fact, that $(x^{(n_k,m_k)})_{k\in\mathbb{N}}$ being a weakly convergent sequence must be bounded.

Now suppose by contradiction that $(0) \notin \overline{\Omega}_{\mathrm{w}}$. Then there exists a weak neighbourhood V of $(0) \in \ell^2$ such that $V \subset \ell^2 \setminus \overline{\Omega}_{\mathrm{w}}$. By definition of weak topology, there exist finitely many open sets $U_1, \ldots, U_r \subset \mathbb{R}$ and elements $y^{(1)}, \ldots, y^{(r)} \in \ell^2$, where $y^{(k)} = (y_j^{(k)})_{j\in\mathbb{N}}$ such that

$$V \supset \bigcap_{k=1}^{r} \{x \in \ell^2 \mid (x, y^{(k)})_{\ell^2} \in U_k\} \ni (0).$$

In particular we have $0 \in U_k$ for every $k \in \{1,\ldots,r\}$. Since every U_k is open and r is finite, there exists $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \subset U_k$ for every $k \in \{1,\ldots,r\}$. However, if we fix $n \in \mathbb{N}$ such that $\frac{1}{n}|y_1^{(k)}| < \frac{\varepsilon}{2}$ and then choose $2 \leq m \in \mathbb{N}$ large enough such that $n|y_m^{(k)}| < \frac{\varepsilon}{2}$ for each of the finitely many $k \in \{1,\ldots,r\}$, we have

$$|(x^{(n,m)}, y^{(k)})_{\ell^2}| \leq \tfrac{1}{n}|y_1^{(k)}| + n|y_m^{(k)}| < \varepsilon \qquad \forall k \in \{1,\ldots,r\}$$

which implies $x^{(n,m)} \in V$. As $x^{(n,m)} \in \Omega$, a contradiction to the definition of V arises.

Solution of 9.7: For completeness, we first prove the representation of the convex hull in the statement.

Lemma. *The following representation theorem for convex hulls holds*

$$\operatorname{conv}(A) = \Big\{\sum_{k=1}^{n} \lambda_k x_k \;\Big|\; n \in \mathbb{N},\ x_1,\ldots,x_n \in A,\ \lambda_1,\ldots,\lambda_n \geq 0,\ \sum_{k=1}^{n} \lambda_k = 1\Big\}.$$

Proof. Given the normed space $(X, \|\cdot\|_X)$ and the subset $A \subset X$, let

$$\mathcal{C} := \Big\{\sum_{k=1}^{n} \lambda_k x_k \;\Big|\; n \in \mathbb{N},\ x_1,\ldots,x_n \in A,\ \lambda_1,\ldots,\lambda_n \geq 0,\ \sum_{k=1}^{n} \lambda_k = 1\Big\}.$$

We prove $\operatorname{conv}(A) = \mathcal{C}$ by showing the two inclusions.

"$\subseteq$" Since $A \subset \mathcal{C}$, the inclusion $\operatorname{conv}(A) \subseteq \mathcal{C}$ follows from the definition of convex hull, if we show that $\mathcal{C}$ is convex. In fact, given $0 < t < 1$ we have

$$t\sum_{k=1}^{n} \lambda_k x_k + (1-t)\sum_{k=1}^{m} \lambda'_k x'_k = \sum_{k=1}^{n+m} \mu_k y_k$$

with

$$0 \leq \mu_k := \begin{cases} t\lambda_k & \text{if } k \in \{1,\ldots,n\}, \\ (1-t)\lambda'_{k-n} & \text{if } k \in \{n+1,\ldots,n+m\} \end{cases}$$

$$A \ni y_k := \begin{cases} x_k & \text{if } k \in \{1,\ldots,n\}, \\ x'_{k-n} & \text{if } k \in \{n+1,\ldots,n+m\} \end{cases}$$

and $\mu_1 + \ldots + \mu_{n+m} = t(\lambda_1 + \ldots + \lambda_n) + (1-t)(\lambda'_1 + \ldots + \lambda'_m) = t + (1-t) = 1$.

"$\supseteq$" Let $x_1, \ldots, x_n \in A$ and let $\lambda_1, \ldots, \lambda_n \geq 0$ with $\lambda_1 + \ldots + \lambda_n = 1$. We can assume $\lambda_1 \neq 0$. Since $\operatorname{conv}(A)$ is convex and contains $x_1, x_2 \in A$, and since $\frac{\lambda_1}{\lambda_1+\lambda_2} + \frac{\lambda_2}{\lambda_1+\lambda_2} = 1$,

$$\operatorname{conv}(A) \ni \frac{\lambda_1}{\lambda_1 + \lambda_2} x_1 + \frac{\lambda_2}{\lambda_1 + \lambda_2} x_2 = \frac{\lambda_1 x_1 + \lambda_2 x_2}{\lambda_1 + \lambda_2} =: y_2.$$

For the same reason,

$$\operatorname{conv}(A) \ni \frac{\lambda_1 + \lambda_2}{\lambda_1 + \lambda_2 + \lambda_3} y_2 + \frac{\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3} x_3 = \frac{\lambda_1 x_2 + \lambda_2 x_2 + \lambda_3 x_3}{\lambda_1 + \lambda_2 + \lambda_3} =: y_3.$$

Iterating this procedure, we obtain

$$\operatorname{conv}(A) \ni \frac{\lambda_1 + \ldots + \lambda_{k-1}}{\lambda_1 + \ldots + \lambda_k} y_{k-1} + \frac{\lambda_k}{\lambda_1 + \ldots + \lambda_k} x_k = \frac{\lambda_1 x_1 + \ldots + \lambda_k x_k}{\lambda_1 + \ldots + \lambda_k} =: y_k.$$

for every $k \in \{3, \ldots, n\}$. Since $\lambda_1 + \ldots + \lambda_n = 1$, we have $y_n = \lambda_1 x_1 + \ldots + \lambda_n x_n$ which concludes the proof of $\operatorname{conv}(A) \supseteq \mathcal{C}$.

$\square$

(i) Let $(x_k)_{k\in\mathbb{N}}$ be a sequence in X and let $x \in X$ such that $x_k \xrightarrow{w} x$ as $k \to \infty$. Let $K := \operatorname{conv}(\{x_k \mid k \in \mathbb{N}\})$. By Problem 9.7, $K \subset \overline{K} \subset \overline{K}_{\text{w-seq}} \subset \overline{K}_{\text{w}}$ but since K is convex, the closure $\overline{K}$ with respect to $\|\cdot\|_X$ agrees with the closure $\overline{K}_{\text{w}}$ with respect to the weak topology: $\overline{K} = \overline{K}_{\text{w}}$. Therefore, the assumption that x is in the weak-sequential closure $\overline{K}_{\text{w-seq}} \ni x$ implies $x \in \overline{K}$ and there exists a sequence $(y_n)_{n\in\mathbb{N}}$ in K such that $\|y_n - x\|_X \to 0$ as $n \to \infty$. By the representation theorem for convex hulls, each element $y_n \in K$ must be a convex linear combination of finitely many elements of $\{x_k \mid k \in \mathbb{N}\}$, which concludes the proof.

(ii) Given the normed space $(X, \|\cdot\|_X)$, the convex subsets $A, B \subset X$ and defining $\triangle := \{(s,t) \in \mathbb{R}^2 \mid s + t = 1,\ s, t \geq 0\}$, we claim that

$$\operatorname{conv}(A \cup B) = \mathcal{D} := \bigcup_{(s,t)\in\triangle} (sA + tB)$$

"$\subseteq$" By choosing $(s,t) = (1,0)$ we see $A \subset \mathcal{D}$. Analogously, $B \subset \mathcal{D}$, hence $A \cup B \subset \mathcal{D}$. If $x \in (\operatorname{conv}(A \cup B)) \setminus (A \cup B)$, then the representation theorem for convex hulls implies that x is of the form

$$x = \sum_{k=1}^{j} s_k a_k + \sum_{k=j+1}^{n} t_k b_k,$$

where $0 \leq j \leq n \in \mathbb{N}$, where $a_k \in A$, $s_k \geq 0$ for all $k = 1, \ldots, j$ and $b_k \in B$, $t_k \geq 0$ for every $k = j+1, \ldots, n$, and where $s_1 + \ldots + s_j + t_{j+1} + \ldots + t_n = 1$. Since $x \notin A \cup B$ by assumption, we have

$$s := \sum_{k=1}^{j} s_k > 0, \qquad\qquad t := \sum_{k=j+1}^{n} t_k > 0,$$

with $s+t=1$. Since A and B are both convex by assumption,

$$a := \frac{1}{s}\sum_{k=1}^{j} s_k a_k \in A, \qquad\qquad b := \frac{1}{t}\sum_{k=j+1}^{n} t_k b_k \in B,$$

and we have shown $x = sa + tb \in \mathcal{D}$.

"$\supseteq$" Let $a \in A$ and $b \in B$. Then $a, b \in \operatorname{conv}(A \cup B)$. Since $\operatorname{conv}(A \cup B)$ is convex, we must have $sa + tb \in \operatorname{conv}(A \cup B)$ for every $(s,t) \in \triangle$. This proves $\operatorname{conv}(A \cup B) \supseteq \mathcal{D}$.

Under the assumption that the convex sets A and B are compact, we show now that

$$\mathcal{D} = \bigcup_{(s,t)\in\triangle} (sA + tB)$$

is compact. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}$. Then there exist $a_n \in A$ and $b_n \in B$ as well as $(s_n, t_n) \in \triangle$ such that $x_n = s_n a_n + t_n b_n$ for every $n \in \mathbb{N}$. We argue in 3 steps:

- Since $\triangle$ is compact in $\mathbb{R}^2$, a subsequence $((s_n,t_n))_{n\in\Lambda_1\subset\mathbb{N}}$ converges in $\triangle$.
- Since A is compact in X, a subsequence $(a_n)_{n\in\Lambda_2\subset\Lambda_1}$ converges in A.
- Since B is compact in X, a subsequence $(b_n)_{n\in\Lambda_3\subset\Lambda_2}$ converges in B.

Therefore, the subsequence $(x_n)_{n\in\Lambda_3}$ converges in $\mathcal{D}$ which concludes the proof.