10.1. Project: The weak topology is not metrizable ⚙ 💎💎💎.

Definition. Let (X,τ) be a topological space. Denoting the set of all neighbourhoods of a point $x\in X$ by

$$\mathcal{U}_x=\{U\subset X\mid \exists\, O\in\tau:\ x\in O\subset U\},$$

we say that $\mathcal{B}_x\subset\mathcal{U}_x$ is a *neighbourhood basis* of x in (X,τ) if $\forall\, U\in\mathcal{U}_x\ \ \exists V\in\mathcal{B}_x:\ V\subset U$.

Definition. A topological space (X,τ) is called *metrizable* if there exists a metric (namely a distance function) $d\colon X\times X\to\mathbb{R}$ on X (as defined in Problem 1.1) such that, denoting $B_\varepsilon(x)=\{y\in X\mid d(x,y)<\varepsilon\}$, there holds

$$\tau=\{O\subset X\mid \forall x\in O\ \exists\,\varepsilon>0:\ B_\varepsilon(x)\subset O)\}.$$

(i) Show that any metrizable topology τ satisfies the *first axiom of countability* which means that each point has a *countable* neighbourhood basis.

From now on, let us assume that $(X,\|\cdot\|_X)$ is a normed space and τ_{w} denotes the weak topology on X.

(ii) Prove that

$$\mathcal{B}:=\left\{\bigcap_{k=1}^{n} f_k^{-1}\Big((-\varepsilon,\varepsilon)\Big)\ \middle|\ n\in\mathbb{N},\ f_1,\ldots,f_n\in X^*,\ \varepsilon>0\right\}$$

is a neighbourhood basis of $0\in X$ in (X,τ_{w}).

(iii) Prove the following lemma: Let $f_1,\ldots,f_n\in X^*$ and $f\in X^*$ be given. Let

$$N:=\{x\in X\mid f_1(x)=\ldots=f_n(x)=0\}.$$

Then $f(x)=0$ for every $x\in N$ if and only if $f=\lambda_1f_1+\ldots+\lambda_nf_n$ for some $\lambda_1,\ldots,\lambda_n\in\mathbb{R}$.

(iv) Using (ii) and (iii), show that if (X,τ_{w}) is first countable, then $(X^*,\|\cdot\|_{X^*})$ admits a countable algebraic basis.

(v) Assume that the normed space $(X,\|\cdot\|_X)$ is infinite dimensional and conclude from (i) and (iv) (recalling also that any algebraic basis of a Banach space is either finite or uncountable) that the topological space (X,τ_{w}) is not metrizable.

10.2. Non-compactness ✍.

In each of the Banach spaces below, find a sequence which is bounded but does not have a convergent subsequence.

(i) $\Big(L^p([0,1]),\|\cdot\|_{L^p([0,1])}\Big)$ for $1\le p\le\infty$.

(ii) $(c_0,\|\cdot\|_{\ell^\infty})$ where $c_0\subset\ell^\infty$ is the space of sequences converging to zero.

10.3. Separability ✎. Let $(X, \|\cdot\|_X)$ be a normed space. Prove that the following statements are equivalent.

(i) The normed space $(X, \|\cdot\|_X)$ is separable.

(ii) $B = \{x \in X \mid \|x\|_X \leq 1\}$ is separable.

(iii) $S = \{x \in X \mid \|x\|_X = 1\}$ is separable.

10.4. Quadratic functional on a reflexive space ⧉. Let $(X, \|\cdot\|_X)$ be a reflexive Banach space over $\mathbb{R}$. Given a positive integer n, consider n pairwise distinct points $x_1, \ldots, x_n$ in X and the functional

$$F\colon X \to \mathbb{R}, \qquad F(x) = \sum_{i=1}^{n} \|x - x_i\|_X^2.$$

(i) Prove that the functional F has a global minimum on X, namely the value $\inf_{x \in X} F(x)$ is a real number attained by F at some $\overline{x} \in X$.

(ii) Let us now assume that $(X, \|\cdot\|_X)$ is a Hilbert space (thus $\|\cdot\|_X$ is induced by a scalar product $\langle \cdot, \cdot \rangle_X$). Prove that the minimum $\overline{x} \in X$ is unique, and that $\overline{x}$ belongs to the convex hull K of $\{x_1, \ldots, x_n\}$.

10.5. A class of functionals on a reflexive space ⧉. Let $(X, \|\cdot\|_X)$ be a reflexive Banach space over $\mathbb{R}$. Let $\ell \in X^*$ and for any given real number $\alpha \geq 0$ consider the functional $F_\alpha\colon X \to \mathbb{R}$ given by

$$F_\alpha(x) = \|x\|_X^2 - |\ell(x)|^\alpha.$$

Prove that there exists $\alpha_0 > 0$ (to be explicitly determined) such that:

(i) For any $0 \leq \alpha < \alpha_0$ the functional F_α has a global minimum on X, namely the value $\inf_{x \in X} F_\alpha(x)$ is a real number attained by F_α at some (not necessarily unique) $\overline{x} \in X$.

(ii) For any $\alpha > \alpha_0$ there exist examples of reflexive Banach spaces $(X, \|\cdot\|_X)$ and linear functionals $\ell \in X^*$ such that one has instead $\inf_{x \in X} F_\alpha(x) = -\infty$.

10. Solutions

Solution of 10.1:

(i) Let (X,τ) be a metrizable topological space. Let $d\colon X\times X\to\mathbb{R}$ be a metric inducing the topology τ. Given $x\in X$, we consider

$$\mathcal{B}_x := \{B_{\frac{1}{n}}(x) \mid n\in\mathbb{N}\}.$$

Let U be any neighbourhood of x. Since (X,τ) is metrizable, there exists $\varepsilon>0$ such that $B_\varepsilon(x)\subset U$. Choosing $\mathbb{N}\ni n>\frac{1}{\varepsilon}$, we have $B_{\frac{1}{n}}(x)\subset U$, which shows that $\mathcal{B}_x$ is a neighbourhood basis of x in (X,τ). Since $x\in X$ is arbitrary and $\mathcal{B}_x$ countable, we have verified the first axiom of countability for (X,τ).

(ii) Let $(X,\|\cdot\|_X)$ be a normed space. Let τ_{w} be the weak topology on X. Let $U\subset X$ be any neighbourhood of $0\in X$ in (X,τ_{w}). Then there exists $\Omega\in\tau_{\mathrm{w}}$ such that $0\in\Omega\subset U$. By definition of weak topology, Ω is an arbitrary union and finite intersection of sets of the form $f^{-1}(I)$ for $f\in X^*$ and $I\subset\mathbb{R}$ open. In particular, Ω contains a finite intersection of such sets containing the origin. More precisely, there exist $f_1,\dots,f_n\in X^*$ and open sets $I_1,\dots,I_n\subset\mathbb{R}$ such that

$$\Omega\supset\bigcap_{k=1}^{n} f_k^{-1}(I_k)\ni 0.$$

By linearity $f_k(0)=0\in I_k$ for every $k\in\{1,\dots,n\}$. Since $I_1,\dots,I_n\subset\mathbb{R}$ are open and n finite, there exists $\varepsilon>0$ such that $(-\varepsilon,\varepsilon)\subset I_k$ for every $k\in\{1,\dots,n\}$. Thus,

$$\Omega\supset\bigcap_{k=1}^{n} f_k^{-1}(I_k)\supset\bigcap_{k=1}^{n} f_k^{-1}\big((-\varepsilon,\varepsilon)\big)$$

and we conclude that a neighbourhood basis of $0\in X$ in (X,τ_{w}) is given by

$$\mathcal{B} := \Big\{\bigcap_{k=1}^{n} f_k^{-1}\big((-\varepsilon,\varepsilon)\big) \,\Big|\, n\in\mathbb{N},\ f_1,\dots,f_n\in X^*,\ \varepsilon>0\Big\}.$$

(iii) Let $f_1,\dots,f_n\in X^*$ and $f\in X^*$ be given. Suppose,

$$f(x)=0 \qquad \forall x\in N := \{x\in X \mid f_1(x)=\ldots=f_n(x)=0\} \tag{$*$}$$

Let the linear map $\varphi\colon X\to\mathbb{R}^n$ be defined by

$$\varphi(x)=\big(f_1(x),\dots,f_n(x)\big).$$

Assumption $(*)$ implies $\ker\varphi\subset\ker f$. Let $F\colon X/\ker\varphi\cong\operatorname{im}(\varphi)\subset\mathbb{R}^n\to\mathbb{R}$ be defined by $F([x]):=f(x)$. This is well-defined since $F([x+p])=f(x)+f(p)=f(x)$ for every $p\in\ker\varphi\subset\ker f$. Defining F to be zero on the orthogonal complement of $\operatorname{im}(\varphi)\subset\mathbb{R}^n$, we obtain a linear map $F\colon\mathbb{R}^n\to\mathbb{R}$ satisfying $f=F\circ\varphi$. By the Riesz representation theorem on $\mathbb{R}^n$ we have $F(y_1,\dots,y_n)=\lambda_1y_1+\ldots+\lambda_ny_n$ for some $(\lambda_1,\dots,\lambda_n)\in\mathbb{R}^n$. This implies

$$f(x)=F(\varphi(x))=\lambda_1f_1(x)+\ldots+\lambda_nf_n(x).$$

Conversely, if f is a linear combination of $\{f_1,\dots,f_n\}$, then $f(x)=0$ for every $x\in N$.

(iv) Let $(X, \|\cdot\|_X)$ be a normed space and suppose that (X, τ_{w}) is first countable. Then there exists a countable neighbourhood basis $\{A_\alpha\}_{\alpha\in\mathbb{N}}$ of $0 \in X$ in (X, τ_{w}). Since $\mathcal{B}$ defined in (ii) is also a neighbourhood basis of $0 \in X$ in (X, τ_{w}), we have

$$\forall \alpha \in \mathbb{N} \quad \exists B_\alpha \in \mathcal{B}: \quad B_\alpha \subset A_\alpha.$$

By construction of $\mathcal{B}$, this means that

$$\forall \alpha \in \mathbb{N} \quad \exists n_\alpha \in \mathbb{N},\ f_1^\alpha, \dots, f_{n_\alpha}^\alpha \in X^*,\ \varepsilon_\alpha > 0: \\ B_\alpha := \{x \in X \mid \forall k = 1, \dots, n_\alpha:\ |f_k^\alpha(x)| < \varepsilon_\alpha\} \subset A_\alpha.$$

We claim that every $f \in X^*$ is a finite linear combination of elements in the set

$$\Upsilon := \bigcup_{\alpha\in\mathbb{N}} \{f_k^\alpha \mid k = 1, \dots, n_\alpha\}.$$

Let $f \in X^*$. Then, $\{x \in X \mid |f(x)| < 1\}$ is a neighbourhood of $0 \in X$ in (X, τ_{w}). Consequently, there exists $\alpha \in \mathbb{N}$ such that $A_\alpha \subset \{x \in X \mid |f(x)| < 1\}$. Then, for every $m > 0$ by linearity

$$\{x \in X \mid \forall k = 1, \dots, n_\alpha:\ |f_k^\alpha(x)| < \tfrac{1}{m}\varepsilon_\alpha\} \\ = \tfrac{1}{m}B_\alpha \subset \tfrac{1}{m}A_\alpha \subset \{\tfrac{1}{m}x \in X \mid |f(x)| < 1\} = \{x \in X \mid |f(x)| < \tfrac{1}{m}\}.$$

Taking the intersection over all $m \in \mathbb{N}$, we obtain

$$\{x \in X \mid \forall k = 1, \dots, n_\alpha:\ f_k^\alpha(x) = 0\} \subset \{x \in X \mid f(x) = 0\}.$$

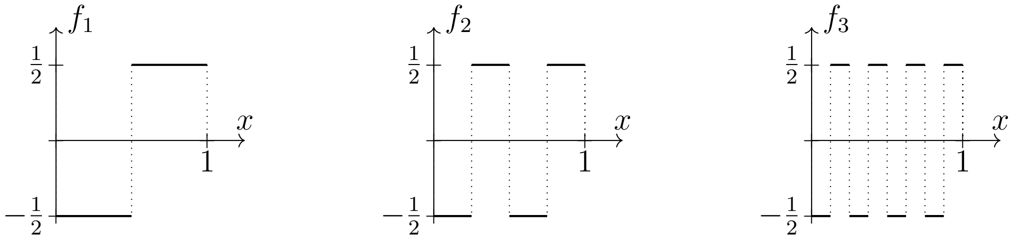
According to part (iii), this implies that f is a linear combination of $\{f_1^\alpha, \dots, f_{n_\alpha}^\alpha\}$ which is a finite subset of Υ. Since $\Upsilon \subset X^*$ is at most countable, we found an algebraic basis for X^* that is at most countable.

(v) Suppose (X, τ_{w}) is metrizable. Then (X, τ_{w}) satisfies the first axiom of countability according to part (i). According to part (iv), an algebraic basis for X^* is at most countable. However, $(X^*, \|\cdot\|_{X^*})$ is always complete because $\mathbb{R}$ is complete (Beispiel 2.1.1). Now recall that any algebraic basis of a complete space is either finite or uncountable, hence in our case the algebrai basis is finite. However, if the algebraic basis of X^* is finite, then $\infty > \dim X^* = \dim X^{**} \geq \dim X$, which contradicts our assumption. Therefore (X, τ_{w}) cannot be metrizable.

Solution of 10.2:

(i) Given $n \in \mathbb{N}$, we divide the interval $[0, 1]$ into 2^n subintervals $I_1, \dots, I_{2^n}$ of equal length, and define the function $f_n\colon [0, 1] \to \mathbb{R}$ on each I_k to be $-\frac{1}{2}$ if k is odd and $+\frac{1}{2}$ if k is even. More precisely,

$$f_n(x) = \begin{cases} -\frac{1}{2}, & \text{if } \exists k \in \mathbb{N}:\ 2^n x \in [2k-2, 2k-1) \\ \frac{1}{2}, & \text{else.} \end{cases}$$



By construction, $\|f_n\|_{L^p([0,1])} = \frac{1}{2}$ for every $n \in \mathbb{N}$ and every $1 \leq p \leq \infty$. Therefore, the sequence $(f_n)_{n\in\mathbb{N}}$ is bounded in $L^p([0,1])$. However by construction, for *any* pair $n, m \in \mathbb{N}$ with $n \neq m$ the difference $|f_n - f_m|$ is the characteristic function of a union of subintervals whose lengths sum up to $\frac{1}{2}$. In particular, $\|f_n - f_m\|_{L^p([0,1])} = (\frac{1}{2})^{\frac{1}{p}}$ for $1 \leq p < \infty$ and $\|f_n - f_m\|_{L^\infty([0,1])} = 1$. Consequently, $(f_n)_{n\in\mathbb{N}}$ cannot have any convergent subsequence.

(ii) Given $n \in \mathbb{N}$, let $e_n \in c_0$ be given by $e_n = (0, \ldots, 0, 1, 0, \ldots)$, where the 1 is at n-th position. Then the sequence $(e_n)_{n\in\mathbb{N}}$ is bounded in $(c_0, \|\cdot\|_{\ell^\infty})$ since $\|e_n\|_{\ell^\infty} = 1$ for every $n \in \mathbb{N}$. However, for *any* pair $n, m \in \mathbb{N}$ with $n \neq m$ we have $\|e_n - e_m\|_{\ell^\infty} = 1$. Consequently, $(e_n)_{n\in\mathbb{N}}$ cannot have any convergent subsequence.

Solution of 10.3: Since subsets of separable sets are separable (Satz 5.2.1), the inclusions $S \subset B \subset X$ already yield (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

Hence, we now conclude the proof by showing that (iii) $\Rightarrow$ (i). By assumption, there exists a countable dense subset $D \subset S$. Moreover, as countable union of countable sets,

$$E := \bigcup_{q\in\mathbb{Q}} qD = \{qd \in X \mid q \in \mathbb{Q},\ d \in D\}$$

is countable. We claim that $E \subset X$ is dense. Let $x \in X$ and $\varepsilon > 0$ be arbitrary. Since $0 \in E$, we may assume $x \neq 0$ and consider the element $\frac{x}{\|x\|_X} \in S$. Since $D \subset S$ is dense, there exists $d \in D$ such that

$$\left\| d - \tfrac{x}{\|x\|_X} \right\|_X < \frac{\varepsilon}{2\|x\|_X}.$$

Moreover, since $\|x\|_X \in \mathbb{R}$ and since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $q \in \mathbb{Q}$ such that

$$\left| q - \|x\|_X \right| < \frac{\varepsilon}{2}.$$

Using $D \subset S \Rightarrow \|d\|_X = 1$ and combining the inequalities, the point $qd \in E$ satisfies

$$\begin{aligned}\|qd - x\|_X &= \left\| (q - \|x\|_X)d + \|x\|_X d - x \right\|_X \\ &\leq \left| q - \|x\|_X \right| + \left\| \|x\|_X d - x \right\|_X < \frac{\varepsilon}{2} + \frac{\varepsilon\|x\|_X}{2\|x\|_X} = \varepsilon,\end{aligned}$$

which proves that $E \subset X$ is dense. Since E is countable, we have shown that X is separable.

Solution of 10.4:

(i) First note that the map F is coercive, because $F(x) \geq \|x - x_1\|_X^2 \to \infty$ as $\|x\|_X \to \infty$. Moreover F is weakly sequentially lower semicontinuous because the map $x \mapsto \|x\|_X$ is.

Hence, since X is reflexive, the direct method (cf. "Variationsprinzip", Satz 5.4.1) applies and we obtain $\overline{x} \in X$ satisfying

$$F(\overline{x}) = \inf_{x \in X} F(x).$$

(ii) Suppose, $\overline{y} \in X \setminus \{\overline{x}\}$ is another minimiser of F and consider $\overline{z} = \frac{1}{2}(\overline{x} + \overline{y})$. Since we are assuming that X is a Hilbert space, the parallelogram identity holds and implies

$$\begin{aligned} \|\overline{z} - x_i\|_X^2 &= \Big\|\frac{\overline{x} - x_i}{2} + \frac{\overline{y} - x_i}{2}\Big\|_X^2 \\ &= 2\Big\|\frac{\overline{x} - x_i}{2}\Big\|_X^2 + 2\Big\|\frac{\overline{y} - x_i}{2}\Big\|_X^2 - \underbrace{\Big\|\frac{\overline{x} - x_i}{2} - \frac{\overline{y} - x_i}{2}\Big\|_X^2}_{\neq 0} \\ &< \frac{\|\overline{x} - x_i\|_X^2}{2} + \frac{\|\overline{y} - x_i\|_X^2}{2}. \end{aligned}$$

Hence, a contradiction follows from

$$F(\overline{z}) < \frac{F(\overline{x})}{2} + \frac{F(\overline{y})}{2} = \inf_{x \in X} F(x)$$

which proves that the minimiser is unique.

Moreover, if $\|\cdot\|_X$ is induced by the scalar product $\langle \cdot, \cdot \rangle_X$, then the minimiser $\overline{x} \in X$ of F has the property that

$$\forall y \in X: \quad 0 = \frac{d}{dt}\Big|_{t=0} F(\overline{x} + ty) = 2\sum_{i=1}^n \langle y, \overline{x} - x_i \rangle_X = 2\Big\langle y, \sum_{i=1}^n (\overline{x} - x_i)\Big\rangle_X.$$

Consequently,

$$\sum_{i=1}^n (\overline{x} - x_i) = 0 \quad \Rightarrow \quad n\overline{x} = \sum_{i=1}^n x_i \quad \Rightarrow \quad \overline{x} = \sum_{i=1}^n \frac{1}{n} x_i$$

which proves that $\overline{x}$ is in the convex hull of $\{x_1, \ldots, x_n\} \subset X$.

Solution of 10.5: We claim that $\alpha_0 = 2$. For every $0 \leq \alpha < 2$, the map F_α is coercive, because

$$|F_\alpha(x)| = \|x\|_X^2 - |\ell(x)|^\alpha \geq \Big(\|x\|_X^{2-\alpha} - \|\ell\|_{X^*}^\alpha\Big)\|x\|_X^\alpha \xrightarrow{\|x\|_X \to \infty} \infty.$$

Moreover F_α is weakly sequentially lower semi-continuous, because $x \mapsto \|x\|_X^2$ is weakly sequentially lower semi-continuous and, by definition of weak convergence, $x \mapsto \ell(x)$ is weakly sequentially continuous.

As a result, since X is reflexive, the direct method (cf. "Variationsprinzip", Satz 5.4.1) applies and we obtain $\bar{x} \in X$ satisfying

$$F_\alpha(\bar{x}) = \inf_{x \in X} F_\alpha(x).$$

Given $\alpha > 2$, consider the example $(X, \|\cdot\|_X) = (\mathbb{R}, |\cdot|)$ (which is reflexive because any finite dimensional Banach space is reflexive) with $\ell(x) = x$. Then

$$F_\alpha(x) = x^2 - |x|^\alpha = \Big(1 - |x|^{\alpha-2}\Big)|x|^2 \xrightarrow{x \to \infty} -\infty.$$