**11.1. Dual operators.** Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be normed spaces. Recall that if $T \in L(X,Y)$, then its dual operator $T^*$ is in $L(Y^*, X^*)$ and it is characterised by the property

$$\forall x \in X \quad \forall y^* \in Y^* : \quad \langle T^* y^*, x\rangle_{X^*\times X} = \langle y^*, Tx\rangle_{Y^*\times Y}.$$

Prove the following facts about dual operators.

(i) $(\mathrm{Id}_X)^* = \mathrm{Id}_{X^*}$

(ii) If $T \in L(X,Y)$ and $S \in L(Y,Z)$, then $(S \circ T)^* = T^* \circ S^*$.

(iii) If $T \in L(X,Y)$ is bijective with inverse $T^{-1} \in L(Y,X)$, then $(T^*)^{-1} = (T^{-1})^*$.

(iv) Let $\mathcal{I}_X \colon X \hookrightarrow X^{**}$ and $\mathcal{I}_Y \colon Y \hookrightarrow Y^{**}$ be the canonical inclusions. Then,

$$\forall\, T \in L(X,Y): \quad \mathcal{I}_Y \circ T = (T^*)^* \circ \mathcal{I}_X.$$

**11.2. Isomorphisms and isometries.** Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces and $T \in L(X,Y)$. Prove the following statements.

(i) If $T$ is an isomorphism, then $T^*$ is an isomorphism.

(ii) If $T$ is an isometric isomorphism, then $T^*$ is an isometric isomorphism.

(iii) If $X$ and $Y$ are both reflexive, then the reverse implications of (i) and (ii) hold.

(iv) If $(X, \|\cdot\|_X)$ is a reflexive Banach space isomorphic to the normed space $(Y, \|\cdot\|_Y)$, then $Y$ is reflexive.

**11.3. Operator on compact sequences.** Consider the space $(c_0, \|\cdot\|_{\ell^\infty})$, where as usual $c_0 := \{(x_n)_{n\in\mathbb{N}} \in \ell^\infty \mid \lim_{n\to\infty} x_n = 0\}$ and the subspace $c_c := \{(x_n)_{n\in\mathbb{N}} \in \ell^\infty \mid \exists N \in \mathbb{N}\ \forall n \geq N:\ x_n = 0\}$. Consider the linear operator

$$T \colon c_c \subset c_0 \to \ell^1, \qquad (Tx)_n = n x_{n+1}.$$

(i) Is $T$ extendable to a bounded linear operator $T\colon c_0 \to \ell^1$? Justify your answer.

(ii) Compute the adjoint of $T$, namely determine

$$T^* \colon D_{T^*} \subset (\ell^1)^* \to (c_0)^*.$$

Notice that the characterization of the subspace $D_{T^*}$ is also required.

(iii) Prove that the operator $T$ is closable. Define the domain $D_{\overline{T}}$ of its closure and determine an element belonging to the set $D_{\overline{T}} \setminus c_c$.

**11.4. Compact operators.** Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be normed spaces. We denote by

$$K(X,Y) = \{T \in L(X,Y) \mid \overline{T(B_1(0))} \subset Y \text{ compact}\}$$

the set of *compact operators* between $X$ and $Y$. Prove the following statements.

(i) $T \in L(X,Y)$ is a compact operator if and only if every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in $X$ has a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $(Tx_{n_k})_{k\in\mathbb{N}}$ is convergent in $Y$.

(ii) If $(Y, \|\cdot\|_Y)$ is complete, then $K(X,Y)$ is a closed subspace of $L(X,Y)$.

(iii) Let $T \in L(X,Y)$. If its range $T(X) \subset Y$ is finite dimensional, then $T \in K(X,Y)$.

(iv) Let $T \in L(X,Y)$ and $S \in L(Y,Z)$. If $T$ or $S$ is a compact operator, then $S \circ T$ is a compact operator.

(v) If $X$ is reflexive, then any operator $T \in L(X,Y)$ which maps weakly convergent sequences to norm-convergent sequences is a compact operator.

**11.5. Integral operators.** Let $m \in \mathbb{N}$ and let $\Omega \subset \mathbb{R}^m$ be a bounded subset. Given $k \in L^2(\Omega \times \Omega)$, consider the linear operator $K\colon L^2(\Omega) \to L^2(\Omega)$ defined by

$$(Kf)(x) = \int_\Omega k(x,y) f(y)\,\mathrm{d}y$$

(i) Prove that $K$ is well-defined, i.e. $Kf \in L^2(\Omega)$ for any $f \in L^2(\Omega)$.

(ii) Prove that $K$ is a compact operator.

**11.6. Operator that is (almost) injective.** Suppose that $X, Y, Z$ are Banach spaces over $\mathbb{R}$, let $P \in L(X,Y)$ and assume that there exists a compact map $J \in L(X,Z)$.

Suppose also that there is a constant $C > 0$ such that for all $x \in X$ one has

$$\|x\|_X \leq C\Big(\|Px\|_Y + \|Jx\|_Z\Big) \tag{$*$}$$

(i) If $P$ is injective, show that there is another constant $C' > 0$ such that for all $x \in X$ one has

$$\|x\|_X \leq C'\|Px\|_Y.$$

(ii) Without assuming that $P$ is injective show that $(*)$ implies that $\ker(P)$ has finite dimension. Hence, prove the existence of a closed subspace $W$ of $X$ with $X = \ker(P) \oplus W$ (i.e. a topologically complementary subspace $W$ of $\ker(P)$ in $X$). Then exploit part (i) to show that for all $x \in W$ one has

$$\|x\|_X \leq C''\|Px\|_Y$$

for some constant $C'' > 0$.

# 11. Solutions

**Solution of 11.1:**

(i) Let $x \in X$ and $x^* \in X^*$ be arbitrary. By definition of $(\mathrm{Id}_X)^*\colon X^* \to X^*$, we have

$$\Big\langle (\mathrm{Id}_X)^* x^*, x \Big\rangle_{X^* \times X} = \Big\langle x^*, \mathrm{Id}_X\, x \Big\rangle_{X^* \times X} = \langle x^*, x \rangle_{X^* \times X^*}.$$

Since $x \in X$ is arbitrary, $(\mathrm{Id}_X)^* x^* = x^*$. Since $x^* \in X^*$ is arbitrary, $(\mathrm{Id}_X)^* = \mathrm{Id}_{(X^*)}$.

(ii) Let $z^* \in Z^*$ and $x \in X$ be arbitrary. Then, $(S \circ T)^* = T^* \circ S^*$ follows from

$$\begin{aligned} \Big\langle (S \circ T)^* z^*, x \Big\rangle_{X^* \times X} &= \Big\langle z^*, S(Tx) \Big\rangle_{Z^* \times Z} \\ &= \Big\langle S^* z^*, Tx \Big\rangle_{Y^* \times Y} = \Big\langle T^*(S^* z^*), x \Big\rangle_{X^* \times X}. \end{aligned}$$

(iii) To prove $(T^*)^{-1} = (T^{-1})^*$, we apply the results from (i) and (ii) and obtain

$$\begin{aligned} T^* \circ (T^{-1})^* &= (T^{-1} \circ T)^* = (\mathrm{Id}_X)^* = \mathrm{Id}_{X^*}, \\ (T^{-1})^* \circ T^* &= (T \circ T^{-1})^* = (\mathrm{Id}_Y)^* = \mathrm{Id}_{Y^*}\,. \end{aligned}$$

(iv) Let $x \in X$ and $y^* \in Y^*$ be arbitrary. Then, $(\mathcal{I}_Y \circ T) = (T^*)^* \circ \mathcal{I}_X$ follows from

$$\begin{aligned} \Big\langle (\mathcal{I}_Y \circ T)x, y^* \Big\rangle_{Y^{**} \times Y^*} &= \Big\langle Tx, y^* \Big\rangle_{Y \times Y^*} = \Big\langle x, T^* y^* \Big\rangle_{X \times X^*} \\ &= \Big\langle \mathcal{I}_X x, T^* y^* \Big\rangle_{X^{**} \times X^*} = \Big\langle (T^*)^*(\mathcal{I}_X x), y^* \Big\rangle_{Y^{**} \times Y^*}. \end{aligned}$$

**Solution of 11.2:**

(i) The dual operator $T^*$ of any $T \in L(X, Y)$ is invertible according to Problem 11.1 (iii) and its inverse is $(T^*)^{-1} = (T^{-1})^*$. Moreover, the assumption $T^{-1} \in L(Y, X)$ implies $(T^{-1})^* \in L(X^*, Y^*)$. Hence, $T^*$ is an isomorphism.

(ii) If $T$ is an isometric isomorphism, then $T^*$ is an isomorphism by (i) and

$$\|T^* y^*\|_{X^*} = \sup_{\|x\|_X \leq 1} \Big| \langle T^* y^*, x \rangle_{X^* \times X} \Big| = \sup_{\|Tx\|_Y = \|x\|_X \leq 1} \Big| \langle y^*, Tx \rangle_{Y^* \times Y} \Big| = \|y^*\|_{Y^*}.$$

(iii) If $X$ and $Y$ are reflexive, $\mathcal{I}_X\colon X \to X^{**}$ and $\mathcal{I}_Y\colon Y \to Y^{**}$ are bijective isometries. If $T^*$ is an (isometric) isomorphism, then (i) and (ii) imply that $(T^*)^*$ is an (isometric) isomorphism. Applying the result of Problem 11.1 (iv), we see that the same holds for

$$T = \mathcal{I}_Y^{-1} \circ (T^*)^* \circ \mathcal{I}_X.$$

(iv) Since $X$ is reflexive by assumption, $\mathcal{I}_X$ is an isomorphism. Suppose, $T\colon X \to Y$ is an isomorphism. Applying part (i) twice, $(T^*)^*$ is an isomorphism. Moreover,

$$\mathcal{I}_Y = (T^*)^* \circ \mathcal{I}_X \circ T^{-1}$$

according to Problem 11.1 (iv). Since $\mathcal{I}_Y$ is a composition of isomorphisms, $Y$ is reflexive.

**Solution of 11.3:**

(i) The operator $T$ is not extendable to a bounded linear operator $T\colon c_0 \to \ell^1$. In fact, denoting $e_k = (0,\ldots,0,1,0,\ldots,0) \in c_c$ (where the 1 is at $k$-th position), we have for all $k \in \mathbb{N}$

$$\|Te_k\|_{\ell^1} = k-1 = (k-1)\|e_k\|_{\ell^\infty}.$$

(ii) Since $(\ell^1)^* \cong \ell^\infty$ and $(c_0)^* \cong \ell^1$ (compare Problem 8.1) we have

$$D_{T^*} = \{y \in \ell^\infty \mid c_c \ni x \mapsto \sum_{n\in\mathbb{N}} y_n (Tx)_n \text{ is continuous}\}.$$

Fixed $y \in \ell^\infty$, the map $A\colon (c_c, \|\cdot\|_{\ell^\infty}) \to \mathbb{R}$ given by

$$Ax = \sum_{n\in\mathbb{N}} y_n (Tx)_n = \sum_{n=0}^{\infty} y_n n x_{n+1}$$

is continuous if

$$\sum_{n\in\mathbb{N}} |ny_n| < \infty$$

because

$$|Ax| = \Big|\sum_{n\in\mathbb{N}} ny_n x_{n+1}\Big| \le \|x\|_{\ell^\infty} \sum_{n\in\mathbb{N}} |ny_n|.$$

Conversely, if $A$ is continuous, we consider $x^{(N)} = (x^{(N)}_n)_{n\in\mathbb{N}} \in c_c$ with $x^{(N)}_n = \frac{y_{n-1}}{|y_{n-1}|}$ for $1 \le n \le N$ and $x^{(N)}_n = 0$ for $n > N$ and $n = 0$ to obtain

$$\|A\| = \|A\|\|x^{(N)}\| \ge |Ax^{(N)}| = \Big|\sum_{n=0}^{N-1} |ny_n|\Big|.$$

Since $N \in \mathbb{N}$ is arbitrary, we conclude

$$\sum_{n\in\mathbb{N}} |ny_n| < \infty.$$

Hence, $D_{T^*} = \{y \in \ell^\infty \mid \sum_{n\in\mathbb{N}} |ny_n| < \infty\}$ and

$$(T^*y)_n = \begin{cases} (n-1)y_{n-1} & (n \ge 1), \\ 0 & (n = 0). \end{cases}$$

(iii) The operator $T$ is closable. Indeed, suppose $x^{(k)} \in c_c$ for $k \in \mathbb{N}$ satisfy

$$\|x^{(k)}\|_{\ell^\infty} \to 0, \qquad \|Tx^{(k)} - y\|_{\ell^1} \to 0 \qquad (k \to \infty)$$

for some $y \in \ell^1$. For every fixed $n \in \mathbb{N}$ in particular,

$$x_n^{(k)} \to 0, \qquad n x_n^{(k)} \to y_n \qquad (k \to \infty)$$

which implies $y_n = 0$ for all $n \in \mathbb{N}$. Hence, $T$ is closable.

Moreover, by definition,

$$D_{\overline{T}} = \{x \in c_0 \mid \exists (x^{(k)})_{k\in\mathbb{N}} \subset c_c,\ y \in \ell^1 :\ (x^{(k)}, Tx^{(k)}) \to (x, y)\}.$$

Consider $x = (n^{-3})_{n\in\mathbb{N}} \in c_0 \setminus c_c$ and $y = (n^{-2})_{n\in\mathbb{N}} \in \ell^1$. Let $x^{(k)} \in c_c$ be the truncation of $x$ at index $k$. Then, $x^{(k)} \to x$ in $c_0$ and

$$\|Tx^{(k)} - y\|_{\ell^1} = \sum_{n=k}^{\infty} n^{-2} \xrightarrow{k\to\infty} 0.$$

Therefore, $x \in D_{\overline{T}}$.

**Solution of 11.4:**

(i) We prove the two implications separately.

**"⇒"** Let $T \in L(X, Y)$ be a compact operator. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in $X$. Then there exists $M > 0$ such that $\|x_n\|_X < M$ for all $n \in \mathbb{N}$. In particular, $\frac{1}{M} x_n \in B_1(0) \subset X$ and $\frac{1}{M} T x_n \in T(B_1(0))$ for every $n \in \mathbb{N}$. Since $\overline{T(B_1(0))} \subset Y$ is compact, a subsequence $(\frac{1}{M} T x_{n_k})_{k\in\mathbb{N}}$ converges in $Y$. Hence, $(T x_{n_k})_{k\in\mathbb{N}}$ also converges.

**"⇐"** Conversely, let $(y_n)_{n\in\mathbb{N}}$ be any sequence in $\overline{T(B_1(0))}$. For every $n \in \mathbb{N}$ there exists $y_n' \in T(B_1(0))$ such that $\|y_n - y_n'\|_Y \le \frac{1}{n}$. Since there exists a sequence $(x_n')_{n\in\mathbb{N}}$ in $B_1(0) \subset X$ such that $T x_n' = y_n'$, a subsequence $y_{n_k}' \to y$ converges in $Y$ as $k \to \infty$ by assumption. Since $\|y_{n_k} - y\|_Y \le \|y_{n_k} - y_{n_k}'\| + \|y_{n_k}' - y\|_Y \to 0$ as $k \to \infty$ we conclude that a subsequence of $(y_n)_{n\in\mathbb{N}}$ converges. Being closed, $\overline{T(B_1(0))}$ must contain the limit $y$, which proves that $\overline{T(B_1(0))}$ is compact, i.e. $T$ is a compact operator.

(ii) Part (i) and linearity of the limit imply that the set of compact operators $K(X, Y) \subset L(X, Y)$ is a linear subspace. To prove that this subspace is closed, let $(T_k)_{k\in\mathbb{N}}$ be a sequence in $K(X, Y)$ such that $\|T_k - T\|_{L(X,Y)} \to 0$ for some $T \in L(X, Y)$ as $k \to \infty$. To show $T \in K(X, Y)$, consider a bounded sequence $(x_n)_{n\in\mathbb{N}}$ in $X$ and choose the nested, unbounded subsets $\mathbb{N} \supset \Lambda_1 \supseteq \Lambda_2 \supseteq \ldots$ such that $(T_k x_n)_{n\in\Lambda_k}$ is convergent in $Y$ with limit point $y_k \in Y$. This is possible by (i), since $T_k$ is a compact operator for every $k \in \mathbb{N}$. Let $\Lambda \subset \mathbb{N}$ be the corresponding diagonal sequence (i.e. the $k$-th number in $\Lambda$ is the $k$-th number in $\Lambda_k$). By continuity of $\|\cdot\|_Y$, we can estimate

$$\|y_k - y_m\|_Y = \lim_{\Lambda\ni n\to\infty} \|T_k x_n - T_m x_n\|_Y \le \|T_k - T_m\|_{L(X,Y)} \sup_{n\in\Lambda} \|x_n\|_X$$

for any $k, m \in \mathbb{N}$. Since $(T_k)_{k\in\mathbb{N}}$ is convergent in $L(X, Y)$, we conclude that $(y_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $Y$. Since $(Y, \|\cdot\|_Y)$ is assumed to be complete, $y_k \to y$ for some $y \in Y$

as $k \to \infty$. We claim that $(Tx_n)_{n\in\Lambda}$ also converges to $y$ which would finish the proof of $T \in K(X,Y)$ by (i). Let $\varepsilon > 0$ and choose a fixed $\kappa \in \mathbb{N}$ such that

$$\|T - T_\kappa\|_{L(X,Y)} < \frac{\varepsilon}{3 \sup\limits_{n\in\Lambda} \|x_n\|_X}, \qquad \|y_\kappa - y\|_Y \leq \frac{\varepsilon}{3}.$$

Since $T_\kappa x_n \to y_\kappa$ as $\Lambda \ni n \to \infty$, there exists $N \in \Lambda$ such that for every $\Lambda \ni n \geq N$ $\|T_\kappa x_n - y_\kappa\| \leq \frac{\varepsilon}{3}$. Finally, the claim follows from the estimate

$$\begin{aligned}\|Tx_n - y\|_Y &\leq \|Tx_n - T_\kappa x_n\|_Y + \|T_\kappa x_n - y_\kappa\|_Y + \|y_\kappa - y\|_Y \\ &\leq \|T - T_\kappa\|_{L(X,Y)} \sup_{n\in\Lambda} \|x_n\|_X + \|T_\kappa x_n - y_\kappa\|_Y + \|y_\kappa - y\|_Y < \varepsilon\end{aligned}$$

which holds for every $\Lambda \ni n \geq N$.

(iii) The image of $B_1(0)$ under $T \in L(X,Y)$ is bounded. If $T(X) \subset Y$ is of finite dimension, then $\overline{T(B_1(0))}$ is compact as a bounded, closed subset of $T(X)$.

(iv) Let $T \in L(X,Y)$ and $S \in L(Y,Z)$. Let $(x_n)_{n\in\mathbb{N}}$ be any bounded sequence in $X$.

Suppose $T$ is a compact operator. Then, a subsequence $(Tx_{n_k})_{k\in\mathbb{N}}$ is convergent in $Y$ by (i). Since $S$ is continuous, $(STx_{n_k})_{k\in\mathbb{N}}$ is convergent in $Z$, which by (i) proves that $S \circ T$ is a compact operator.

Suppose $S$ is a compact operator. Since $T$ is continuous, the sequence $(Tx_n)_{n\in\mathbb{N}}$ is bounded in $Y$. Then, a subsequence $(STx_{n_k})_{k\in\mathbb{N}}$ is convergent in $Z$ by (i), which again proves that $S \circ T$ is a compact operator.

(v) Let $(x_n)_{n\in\mathbb{N}}$ be any bounded sequence in $X$. Since $X$ is reflexive, a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ converges weakly in $X$ by the Eberlein–Šmulian theorem. Then, $(Tx_{n_k})_{k\in\mathbb{N}}$ is norm-convergent in $Y$ by assumption and (i) implies that $T$ is a compact operator.

**Solution of 11.5:**

(i) Let $f \in L^2(\Omega)$. Then Hölder's inequality and Fubini's theorem imply

$$\begin{aligned}\int_\Omega |(Kf)(x)|^2 \,\mathrm{d}x &= \int_\Omega \Big| \int_\Omega k(x,y) f(y) \,\mathrm{d}y \Big|^2 \mathrm{d}x \leq \int_\Omega \Big( \int_\Omega |k(x,y) f(y)| \,\mathrm{d}y \Big)^2 \mathrm{d}x \\ &\leq \int_\Omega \Big( \int_\Omega |k(x,y)|^2 \,\mathrm{d}y \Big) \|f\|^2_{L^2(\Omega)} \,\mathrm{d}x = \|k\|^2_{L^2(\Omega\times\Omega)} \|f\|^2_{L^2(\Omega)}.\end{aligned}$$

Since $k \in L^2(\Omega \times \Omega)$ by assumption, $\|Kf\|_{L^2(\Omega)} \leq \|k\|_{L^2(\Omega\times\Omega)} \|f\|_{L^2(\Omega)} < \infty$ follows.

(ii) Since the space $L^2(\Omega)$ is reflexive (which follows from being a Hilbert space), Problem 11.4 (v) implies that $K \colon L^2(\Omega) \to L^2(\Omega)$ is a compact operator, if $K$ maps weakly convergent sequences to norm-convergent sequences.

Let $(f_n)_{n\in\mathbb{N}}$ be sequence in $L^2(\Omega)$ such that $f_n \xrightarrow{w} f$ as $n \to \infty$ for some $f \in L^2(\Omega)$. Since $k \in L^2(\Omega \times \Omega)$, Fubini's theorem implies that $k(x,\cdot) \in L^2(\Omega)$ for almost every $x \in \Omega$. Weak convergence therefore implies

$$(Kf_n)(x) = \big\langle k(x,\cdot), f_n \big\rangle_{L^2(\Omega)} \xrightarrow{n\to\infty} \big\langle k(x,\cdot), f \big\rangle_{L^2(\Omega)} = (Kf)(x)$$

for almost every $x \in \Omega$. As weakly convergent sequence, $(f_n)_{n\in\mathbb{N}}$ is bounded: there exists $C \in \mathbb{R}$ such that $\|f_n\|_{L^2(\Omega)} \leq C$ for every $n \in \mathbb{N}$. By Hölder's inequality,

$$|(Kf_n)(x)| \leq \int_\Omega |k(x,y)f_n(y)|\,\mathrm{d}y \leq \|k(x,\cdot)\|_{L^2(\Omega)}\|f_n\|_{L^2(\Omega)} \leq C\|k(x,\cdot)\|_{L^2(\Omega)}.$$

The assumption $k \in L^2(\Omega\times\Omega)$ and Fubini's theorem imply that the function $x \mapsto C\|k(x,\cdot)\|_{L^2(\Omega)}$ is in $L^2(\Omega)$. Thus, $(Kf_n)(x)$ is dominated by a function in $L^2(\Omega)$. Since $(Kf_n)(x)$ converges pointwise for almost every $x \in \Omega$ to a function in $L^2(\Omega)$, the dominated convergence theorem implies $L^2$-convergence $\|Kf_n - Kf\|_{L^2(\Omega)} \to 0$.

**Solution of 11.6:**

(i) For the sake of a contradiction, assume the claimed inequality is false: thus for any $k \geq 1$ one can find $x_k \in X$ with $\|x_k\|_X = 1$ and $\|Px_k\|_Y \leq \frac{1}{k}$. By compactness of the map $J\colon X \to Z$ one can find $\Lambda \subset \mathbb{N}$ such that

$$Jx_k \to z_\infty \ \text{ in } \ (Z, \|\cdot\|_Z) \ \ (k\to\infty,\ k\in\Lambda).$$

At this stage, using $(*)$ with $x_l - x_m$ in lieu of $x$, namely

$$\|x_l - x_m\|_X \leq C\Big(\|P(x_l-x_m)\|_Y + \|J(x_l-x_m)\|_Z\Big).$$

one gets that the sequence $(x_k)_{k\in\Lambda}$ is Cauchy in $(X,\|\cdot\|_X)$ so by completeness $x_k \to x_\infty$ in $(X,\|\cdot\|_X)$ $(k\to\infty,\ k\in\Lambda)$. Since $P \in L(X,Y)$ we have

$$x_k \to x_\infty \implies Px_k \to Px_\infty \ \ (k\to\infty,\ k\in\Lambda)$$

but one the other hand $Px_k \to 0$ by construction, so we conclude $Px_\infty = 0$ and hence, by injectivity $x_\infty = 0$. However it should be $\|x_\infty\|_X = 1$ by the fact that $\|x_k\|_X = 1$ for any $k \geq 1$, contradiction.

(ii) Let us prove that $\ker(P)$ has finite dimension by showing that $B_1(0;\ker(P))$ is relatively compact in $(X,\|\cdot\|_X)$. To this scope, pick $(x_k)_{k\in\mathbb{N}} \subset \ker(P)$ a sequence with $\|x_k\|_X < 1$ and let us prove it has a converging subsequence. Observe that inequality $(*)$, when restricted to $x \in \ker(P)$ takes the form

$$\|x\|_X \leq C\|Jx\|_Z.$$

Hence (arguing as above) one first gets $Jx_k \to z_\infty$ $(k\to\infty, k\in\Lambda)$, by compactness of $J$, and then, by the inequality above, $(x_k)_{k\in\Lambda}$ is Cauchy in $(X,\|\cdot\|_X)$ hence convergent to $x_\infty$.

At this stage, the fact that $\ker(P)$ is topologically complemented in $X$ follows by Problem 7.2, so let us write $X = \ker(P) \oplus W$ with $W \subset X$ closed (i.e. $\overline{W} = W$) by Problem 3.4.

Lastly, the restricted operator $P^\rho\colon W \to Y$ is linear, bounded and one can invoke the result of part (i). With $W$ in lieu of $X$ and $P^\rho$ in lieu of $P$ to conclude that $\|x\|_X \leq C''\|Px\|_Z$ uniformly for $x \in W \subset X$, which completes the proof.