13.1. Definitions of resolvent set ✍. Let $(X, \|\cdot\|_X)$ be a Banach space over $\mathbb{C}$ and let $A\colon D_A \subset X \to X$ be a linear operator. Prove that if A has closed graph, then the following sets coincide.

$$\rho(A) = \{\lambda \in \mathbb{C} \mid (\lambda - A)\colon D_A \to X \text{ is bijective, } \exists(\lambda - A)^{-1} \in L(X, X)\},$$

$$\tilde{\rho}(A) = \{\lambda \in \mathbb{C} \mid (\lambda - A)\colon D_A \to X \text{ is injective with dense image, } \exists(\lambda - A)^{-1} \in L(Z(\lambda), X)\},$$

where we have set $Z(\lambda) := (\lambda - A)(D_A)$, i.e., the image of $\lambda - A$, and $(\lambda - A)^{-1} \in L(Z(\lambda), X)$ means that the (necessarily linear) set-theoretic inverse of $\lambda - A$ is bounded, in the usual sense that $\sup_{z \in Z_\lambda, \|z\|_X \leq 1} \|(\lambda - A)^{-1}(z)\| < \infty$.

Remark. In the literature, the resolvent set is often defined to be $\tilde{\rho}(A)$ rather than $\rho(A)$. Since as soon as $\rho(A)$ is not empty A has closed graph, this problem shows that the two perspectives are in fact equivalent.

13.2. Unitary operators ✍.

Definition. Let $(H, \langle\cdot,\cdot\rangle_H)$ be a Hilbert space over $\mathbb{C}$. An invertible linear operator $T \in L(H, H)$ is called *unitary*, if $T^* = T^{-1}$.

(i) Prove that $T \in L(H, H)$ is unitary if and only if T is a bijective isometry.

(ii) Prove that if $T \in L(H, H)$ is unitary, then its spectrum lies on the unit circle:

$$\sigma(T) \subset \mathbb{S}^1 := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}.$$

13.3. Integral operators revisited ✍. Let $\Omega \subset \mathbb{R}^m$ be a bounded subset. Given $k \in L^2(\Omega \times \Omega)$ such that $k(x, y) = k(y, x)$ for almost every $(x, y) \in \Omega \times \Omega$, consider the operator $K\colon L^2(\Omega) \to L^2(\Omega)$ defined by

$$(Kf)(x) = \int_\Omega k(x, y) f(y) \, \mathrm{d}y$$

and the operator

$$A\colon L^2(\Omega) \to L^2(\Omega)$$
$$f \mapsto f - Kf.$$

Prove that injectivity of A and surjectivity of A are equivalent.

13.4. Resolvents and spectral distance ⚙💎💎. Let $(H, \langle\cdot,\cdot\rangle_H)$ be a Hilbert space over $\mathbb{C}$.

(i) Let $A \in L(H, H)$ be a self-adjoint operator and let $\lambda \in \rho(A)$ be an element in its resolvent set. Show that the resolvent $R_\lambda := (\lambda - A)^{-1}$ is a *normal* operator, i.e.,

$$R_\lambda R_\lambda^* = R_\lambda^* R_\lambda.$$

(ii) Let $A, B \in L(H,H)$ be self-adjoint operators. The *Hausdorff distance* of their spectra $\sigma(A), \sigma(B) \subset \mathbb{C}$ is defined to be

$$d\Big(\sigma(A), \sigma(B)\Big) := \max\left\{ \sup_{\alpha\in\sigma(A)} \Big(\inf_{\beta\in\sigma(B)} |\alpha-\beta| \Big), \sup_{\beta\in\sigma(B)} \Big(\inf_{\alpha\in\sigma(A)} |\alpha-\beta| \Big) \right\}.$$

Prove the estimate

$$d\Big(\sigma(A), \sigma(B)\Big) \leq \|A-B\|_{L(H,H)}.$$

Remark. The Hausdorff distance d is in fact a distance on compact subsets of $\mathbb{C}$. In particular, it restricts to an actual distance function on the spectra of bounded linear operators.

13.5. Compact operator on space decomposition. Let H be a Hilbert space over $\mathbb{R}$ and let $A\colon H \to H$ be linear, compact and self-adjoint.

(i) State the spectral theorem for A.

Now, suppose the existence of two complementary and mutually orthogonal subspaces $H', H'' \subset H$ that are A-invariant, meaning that

$$H = H' \oplus^{\perp} H'', \qquad A(H') \subset H', \qquad A(H'') \subset H''.$$

(ii) Show that each of the restricted operators $A' := A_{|H'}$ and $A'' := A_{|H''}$ is also compact and self-adjoint.

Assume now that A is nonnegative definite (i.e., $(Ax, x) \geq 0$ for all $x \in H$).

(iii) State the Courant–Fischer characterization of the eigenvalues of A.

(iv) Denoted by $\lambda_1, \lambda_1', \lambda_1''$ the first (namely, the *largest*) eigenvalue of A, A', A'' respectively, show that

$$\lambda_1 = \max\{\lambda_1', \lambda_1''\}.$$

13.6. Heisenberg's uncertainty principle. Let $(H, \langle\cdot,\cdot\rangle_H)$ be a Hilbert space over $\mathbb{C}$. Let $D_A, D_B \subset H$ be dense subspaces and let $A\colon D_A \subset H \to H$ and $B\colon D_B \subset H \to H$ be symmetric linear operators. Under the necessary assumption that $A(D_A \cap D_B) \subset D_B$ and $B(D_A \cap D_B) \subset D_A$, the *commutator*

$$\begin{aligned} [A,B]\colon D_{[A,B]} \subset H &\to H \\ x &\mapsto A(Bx) - B(Ax) \end{aligned}$$

is a well-defined operator on $D_{[A,B]} := D_A \cap D_B$.

(i) Prove the following inequality:

$$\forall x \in D_{[A,B]}: \qquad 2\|Ax\|_H \|Bx\|_H \geq \Big|\langle x, [A,B]x\rangle_H\Big|.$$

(ii) Given the symmetric operator $A\colon D_A \subset H \to H$ we define the *standard deviation*

$$\varsigma(A,x) := \sqrt{\langle Ax, Ax\rangle_H - \langle x, Ax\rangle_H^2}$$

at each $x \in D_A$ with $\|x\|_H = 1$. Verify $\varsigma(A,x) \in \mathbb{R}$ and prove the following inequality:

$$\forall x \in D_{[A,B]},\ \|x\|_H = 1: \quad 2\varsigma(A,x)\,\varsigma(B,x) \geq \Big|\langle x, [A,B]x\rangle_H\Big|.$$

Remark. The possible *states* of a quantum mechanical system are given by elements $x \in H$ with $\|x\|_H = 1$. Each *observable* is given by a symmetric linear operator $A\colon D_A \subset H \to H$. If the system is in state $x \in D_A$, we measure the observable A with uncertainty $\varsigma(A,x)$.

(iii) Let $A\colon D_A \to H$ and $B\colon D_B \to H$ be as above. The pair of operators (A,B) is called *Heisenberg pair* if

$$[A,B] = i\,\mathrm{id}_{D_{[A,B]}}.$$

Under the assumption that B has finite operator norm and $D_B = H$, prove that if (A,B) is a Heisenberg pair, then $A\colon D_A \subset H \to H$ cannot have finite operator norm.

(iv) Consider the Hilbert space $(H, \langle\cdot,\cdot\rangle_H) = \Big(L^2([0,1];\mathbb{C}), \langle\cdot,\cdot\rangle_{L^2}\Big)$ and the subspace

$$C_0^1([0,1];\mathbb{C}) := \{f \in L^2([0,1];\mathbb{C}) \mid f \in C^1([0,1];\mathbb{C}),\ f(0) = 0 = f(1)\}.$$

Here, we denote elements in the Hilbert space $L^2([0,1];\mathbb{C})$ by f and points in the interval $[0,1]$ by s. We understand $f \in C^1([0,1];\mathbb{C})$ if f has a representative in C^1 and write $f' = \frac{\mathrm{d}}{\mathrm{d}s}f$ in this case. Recall that in this sense, $C_0^1([0,1];\mathbb{C}) \subset L^2([0,1];\mathbb{C})$ is a dense subspace. The operators

$$P\colon C_0^1([0,1];\mathbb{C}) \to L^2([0,1];\mathbb{C}), \quad f(s) \mapsto if'(s) \qquad\qquad Q\colon L^2([0,1];\mathbb{C}) \to L^2([0,1];\mathbb{C}), \quad f(s) \mapsto sf(s)$$

correspond to the observables *momentum* and *position*. Check that P and Q are well-defined, symmetric operators. Check that $[P,Q]\colon C_0^1([0,1];\mathbb{C}) \to L^2([0,1];\mathbb{C})$ is well-defined.

Show that (P,Q) is a Heisenberg pair and conclude the uncertainty principle:

$$\forall f \in C_0^1([0,1];\mathbb{C}),\ \|f\|_{L^2([0,1];\mathbb{C})} = 1: \quad \varsigma(P,f)\,\varsigma(Q,f) \geq \tfrac{1}{2}.$$

The more precisely the momentum of some particle is known, the less precisely its position can be known, and vice versa.

13. Solutions

Solution of 13.1: Let $\lambda \in \tilde{\rho}(A)$. To show $\lambda \in \rho(A)$, we need to prove that $(\lambda - A)\colon D_A \to X$ is surjective. Let $y \in X$. Since $(\lambda - A)$ has dense image, there exists a sequence $(y_n)_{n\in\mathbb{N}}$ in the image $Z(\lambda)$ of $(\lambda - A)$ such that $\|y_n - y\|_X \to 0$ as $n \to \infty$. Let $x_n = (\lambda - A)^{-1} y_n \in D_A$. Since $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in Y, and since

$$\|x_m - x_n\|_X = \|(\lambda - A)^{-1}(y_m - y_n)\|_X \leq \|(\lambda - A)^{-1}\|_{L(Z(\lambda),X)} \|y_m - y_n\|_X,$$

we conclude that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Since X is complete, there exists a limit point $X \ni x = \lim_{n\to\infty} x_n$. Moreover,

$$Ax_n = \lambda x_n - y_n \xrightarrow{n\to\infty} \lambda x - y.$$

Since A has closed graph $x \in D_A$ with $Ax = \lambda x - y$. This implies $y = (\lambda - A)x$. Thus, $(\lambda - A)$ is surjective and $\lambda \in \rho(A)$ follows. The reverse inclusion $\rho(A) \subset \tilde{\rho}(A)$ is trivial.

Solution of 13.2:

(i) Suppose, $T \in L(H,H)$ is a unitary operator. Then, T is invertible with inverse $T^{-1} = T^* \in L(H,H)$. In particular, T is bijective. T is also an isometry, because

$$\forall x \in H: \qquad \|Tx\|_H^2 = \langle Tx, Tx\rangle_H = \langle T^*Tx, x\rangle_H = \langle x, x\rangle_H = \|x\|_H^2.$$

Conversely, suppose, $T \in L(H,H)$ is a bijective isometry. Then, $\|Tx\|_H^2 = \|x\|_H^2$ for every $x \in H$. From the (complex) polarization identity

$$\langle x, y\rangle_H = \frac{1}{4}\Big(\|x+y\|^2 - \|x-y\|^2\Big) + \frac{i}{4}\Big(\|x+iy\|_H^2 - \|x-iy\|_H^2\Big)$$

(which is motivated by the parallelogram identity), we conclude $\langle Tx, Ty\rangle_H = \langle x, y\rangle_H$ for every $x, y \in H$. In other words, isometries preserve the scalar product. Therefore,

$$\langle T^*Tx, y\rangle_H = \langle Tx, Ty\rangle_H = \langle x, y\rangle_H$$

for every $x, y \in H$ which implies $T^*Tx = x$ for every $x \in H$. Since T is bijective, we obtain $T^* = T^{-1}$ which means that T is unitary.

(ii) Let $T \in L(H,H)$ be unitary. Part (i) implies that T and $T^* = T^{-1}$ are bijective isometries. Therefore, $\|T\| = 1 = \|T^*\|$. Since the spectral radius of T is bounded from above by $\|T\| = 1$, we obtain $\{\lambda \in \mathbb{C} \mid |\lambda| > 1\} \subset \rho(T)$ from Satz 6.5.3.i.

Given $\lambda \in \mathbb{C}$ with $0 \leq |\lambda| < 1$, the spectral radius of the operator λT^* is bounded from above by $\|\lambda T^*\| = |\lambda| < 1$. Thus, $(1 - \lambda T^*)$ is invertible on H by Satz 2.2.7. Hence, $(\lambda - T) = -T \circ (1 - \lambda T^*)$ is bijective as composition of bijective operators and we obtain $\lambda \in \rho(T)$. To conclude, $\sigma(T) \subset \mathbb{S}^1$.

Solution of 13.3: From $k(x,y)=k(y,x)$ for almost every $(x,y)\in\Omega\times\Omega$ and with the help of Fubini's theorem, we conclude that the integral operator $K\colon L^2(\Omega)\to L^2(\Omega)$ is symmetric. Indeed we have

$$\begin{aligned}\forall f,g\in L^2(\Omega):\quad \langle Kf,g\rangle_{L^2(\Omega)} &= \int_\Omega\Big(\int_\Omega k(x,y)f(y)\,\mathrm{d}y\Big)g(x)\,\mathrm{d}x\\ &= \int_\Omega f(y)\Big(\int_\Omega k(y,x)g(x)\,\mathrm{d}x\Big)\,\mathrm{d}y = \langle f,Kg\rangle_{L^2(\Omega)}.\end{aligned}$$

In fact, K is self-adjoint, since moreover it holds $D_K=L^2(\Omega)=D_{K^*}$. Therefore, the operator $A=(1-K)\colon L^2(\Omega)\to L^2(\Omega)$ is also self-adjoint (Beispiel 6.4.2.ii).

According to Problem 11.5 (ii), K is a compact operator, which implies that the operator $A=(1-K)$ has closed image $\operatorname{im}(A)\subset H$. According to Banach's closed range theorem, this is equivalent to $\operatorname{im}(A)=\ker(A^*)^\perp$. Since $A^*=A$, we conclude

$$A\text{ surjective}\iff H=\operatorname{im}(A)=\ker(A)^\perp\iff\ker(A)=\{0\}\iff A\text{ injective.}$$

Solution of 13.4:

(i) Given the self-adjoint operator $A\in L(H,H)$ and an element $\lambda\in\rho(A)$, the operator $(\lambda-A)\in L(H,H)$ is bijective with inverse $R_\lambda=(\lambda-A)^{-1}\in L(H,H)$. Problem 11.2 (i) then implies that R_λ^* is an isomorphism and according to Problem 11.1 (iii),

$$R_\lambda^*=\Big((\lambda-A)^{-1}\Big)^*=\Big((\lambda-A)^*\Big)^{-1}=(\overline{\lambda}-A^*)^{-1}=(\overline{\lambda}-A)^{-1}=R_{\overline{\lambda}}.$$

Alternatively, for any $x,y\in H$, we can directly compute

$$\begin{aligned}\langle x,y\rangle_H &= \langle(\lambda-A)R_\lambda x,y\rangle_H=\langle\lambda R_\lambda x,y\rangle_H-\langle AR_\lambda x,y\rangle_H\\ &=\langle R_\lambda x,\overline{\lambda}y\rangle_H-\langle R_\lambda x,Ay\rangle_H=\langle R_\lambda x,(\overline{\lambda}-A)y\rangle_H=\langle x,R_\lambda^*(\overline{\lambda}-A)y\rangle_H\end{aligned}$$

which implies $R_\lambda^*(\overline{\lambda}-A)y=y$ for any $y\in H$. According to Satz 6.5.2, resolvents commute: $R_\lambda R_{\overline{\lambda}}=R_{\overline{\lambda}}R_\lambda$. This implies that R_λ is a normal operator: $R_\lambda R_\lambda^*=R_\lambda^*R_\lambda$.

(ii) Let $A,B\in L(H,H)$ be self-adjoint operators. By symmetry of the Hausdorff distance (in the sense that we can switch the roles of A and B), it suffices to prove

$$\sup_{\alpha\in\sigma(A)}\Big(\inf_{\beta\in\sigma(B)}|\alpha-\beta|\Big)\le\|A-B\|_{L(H,H)}.$$

The claim follows, if we show the following implication for any $\alpha\in\mathbb{C}$:

$$\inf_{\beta\in\sigma(B)}|\alpha-\beta|>\|A-B\|_{L(H,H)}\qquad\Longrightarrow\quad\alpha\in\rho(A)=\mathbb{C}\setminus\sigma(A).$$

Let $\alpha\in\mathbb{C}$ satisfy $\inf_{\beta\in\sigma(B)}|\alpha-\beta|>\|A-B\|_{L(H,H)}$. Since the claim is trivial otherwise, we may assume $\|A-B\|_{L(H,H)}>0$. Then, α has positive distance from $\sigma(B)$ which implies $\alpha\in\rho(B)$. Hence, $(\alpha-B)^{-1}$ is well-defined and we obtain

$$(\alpha-A)=(\alpha-B)-(A-B)=\Big(1-(A-B)(\alpha-B)^{-1}\Big)(\alpha-B).\qquad(*)$$

Since $(\alpha - B)$ is bijective, it remains to prove that $\Big(1 - (A-B)(\alpha-B)^{-1}\Big)$ is bijective. This follows from Satz 2.2.7 if we prove $\|(A-B)(\alpha-B)^{-1}\|_{L(H,H)} < 1$.

Consider the rational function $f_\alpha\colon \mathbb{C} \to \mathbb{C}$ given by $f_\alpha(z) = (\alpha - z)^{-1}$. By assumption,

$$\frac{1}{\|A-B\|} > \frac{1}{\inf\limits_{\beta\in\sigma(B)}|\alpha-\beta|} = \sup_{\beta\in\sigma(B)}\frac{1}{|\alpha-\beta|} = \sup\Big\{|x| \;\Big|\; x\in f_\alpha(\sigma(B))\Big\}.$$

The spectral mapping theorem (Satz 6.5.4) implies $f_\alpha(\sigma(B)) = \sigma(f_\alpha(B))$. Thus,

$$\frac{1}{\|A-B\|} > \sup\Big\{|x| \;\Big|\; x\in \sigma(f_\alpha(B))\Big\} = \sup_{x\in\sigma(f_\alpha(B))}|x| = r_{f_\alpha(B)}, \tag{†}$$

where we use the characterisation of spectral radius proven in Satz 6.5.3. Since $f_\alpha(B) = (\alpha-B)^{-1} =: R$ is a resolvent of B, it is a normal operator by (i). Hence,

$$\begin{aligned}
&\|Rx\|_H^2 = \langle Rx, Rx\rangle_H = \langle R^*Rx, x\rangle_H = \langle RR^*x, x\rangle_H = \langle R^*x, R^*x\rangle_H = \|R^*x\|_H^2,\\
&\|Rx\|_H^2 = \langle R^*Rx, x\rangle_H \le \|R^*Rx\|_H\|x\|_H \le \|R^*R\|\|x\|_H^2,\\
\implies\quad &\|R\|^2 \le \|R^*R\| \le \|R^*\|\|R\| = \|R\|^2,\\
\implies\quad &\|R\|^2 = \|R^*R\| = \sup_{\|x\|_H=1}\|R^*(Rx)\|_H = \sup_{\|x\|_H=1}\|R(Rx)\|_H = \|R^2\|.
\end{aligned}$$

(Note how the last identity makes use of the first identity.) Inductively, we obtain $\|R\|^{2^n} = \|R^{2^n}\|$ for every $n\in\mathbb{N}$ which implies $r_{f_\alpha(B)} = r_R = \|R\| = \|(\alpha-B)^{-1}\|$. Combined with estimate (†), we obtain $\frac{1}{\|A-B\|} > \|(\alpha-B)^{-1}\|$, which yields

$$\|(A-B)(\alpha-B)^{-1}\| \le \|A-B\|\|(\alpha-B)^{-1}\| < 1$$

and proves the claim. From $(*)$ we then conclude $\alpha\in\rho(A)$.

Solution of 13.5:

(i) Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and let $A\colon H\to H$ be linear, compact and self-adjoint and $A\neq 0$. Then there exist at most countably many eigenvalues $\lambda_k\in\mathbb{R}\setminus\{0\}$ which can accumulate only at $0\in\mathbb{R}$ and corresponding eigenvectors $e_k\in H$ such that

$$\forall x\in H:\quad Ax = \sum_k \lambda_k\langle x, e_k\rangle e_k.$$

(ii) As an orthogonal complement of H'' the subspace $H'\subset H$ is closed and $(H', \langle\cdot,\cdot\rangle)$ is Hilbertean. Let $B_1\subset H$ be the unit ball in H and $B_1'\subset H'$ the unit ball in H'. Then, $\overline{A'B_1'} = \overline{AB_1'}$ is compact as closed subset of the compact set $\overline{AB_1}$. Therefore, $A'\colon H'\to H'$ and analogously $A''\colon H''\to H''$ are compact operators.

Moreover, we have

$$\forall x,y\in H':\quad \langle A'x, y\rangle = \langle Ax, y\rangle = \langle x, Ay\rangle = \langle x, A'y\rangle$$

Hence, $A'\colon H'\to H'$ is symmetric and hence self-adjoint being defined on all of H'. Self-adjointness of $A''\colon H''\to H''$ follows analogously.

(iii) The Courant–Fischer characterization of the k-th eigenvalue λ_k of A is

$$\lambda_k = \sup_{\substack{M\subset H,\\ \dim M=k}} \inf_{\substack{x\in M,\\ \|x\|=1}} \langle x, Ax\rangle.$$

(iv) By the Courant–Fischer characterization

$$\lambda_1 = \sup_{\substack{x\in H,\\ \|x\|=1}} \langle x, Ax\rangle \geq \sup_{\substack{x'\in H',\\ \|x'\|=1}} \langle x', Ax'\rangle = \sup_{\substack{x'\in H',\\ \|x'\|=1}} \langle x', A'x'\rangle = \lambda_1'.$$

Analogously, $\lambda_1 \geq \lambda_1''$, hence we have $\lambda_1 \geq \max\{\lambda_1', \lambda_1''\}$.

If $e_1 = e_1' + e_1'' \in H' \oplus H''$ is an eigenvector of A relative to its first eigenvalue $\lambda_1 > 0$, then

$$A'e_1' + A''e_1'' = Ae_1 = \lambda_1 e_1' + \lambda_1 e_1''$$
$$\implies \begin{cases} \langle e_1', A'e_1'\rangle = \lambda_1\langle e_1', e_1'\rangle \\ \langle e_1'', A''e_1''\rangle = \lambda_1\langle e_1'', e_1''\rangle \end{cases}$$

If $e_1' \neq 0$, then $e_1'/\|e_1'\|$ is an eigenvector with eingevalue λ_1 and we obtain $\lambda_1' \geq \lambda_1$. Analogously, if $e_1'' \neq 0$, then $e_1''/\|e_1''\|$ is an eigenvector with eingevalue λ_1 and we obtain $\lambda_1'' \geq \lambda_1$. Since $e_1 \neq 0$, one of these two cases must be true, and we conclude $\lambda_1 = \max\{\lambda_1, \lambda_2\}$.

Solution of 13.6: Let $A\colon D_A \subset H \to H$ and $B\colon D_B \subset H \to H$ be densely defined, symmetric linear operators on the Hilbert space $(H, \langle\cdot,\cdot\rangle_H)$ such that $A(D_A) \subset D_B$ and $B(D_B) \subset D_A$.

(i) Let $x \in D_{[A,B]} := D_A \cap D_B$. Then, applying the Cauchy–Schwarz inequality,

$$\begin{aligned}\Big|\langle x, [A,B]x\rangle_H\Big| &\leq \Big|\langle x, A(Bx)\rangle_H\Big| + \Big|\langle x, B(Ax)\rangle_H\Big| = \Big|\langle Ax, Bx\rangle_H\Big| + \Big|\langle Bx, Ax\rangle_H\Big| \\ &\leq 2\|Ax\|_H\|Bx\|_H.\end{aligned}$$

(ii) Since A is a symmetric operator, $\langle x, Ax\rangle_H$ is real for any $x \in D_A \subset D_{A^*}$. Indeed,

$$\langle x, Ax\rangle_H = \langle A^*x, x\rangle_H = \langle Ax, x\rangle_H = \overline{\langle x, Ax\rangle}_H.$$

Moreover, for $x \in D_A$ with $\|x\|_H = 1$, we have

$$\langle x, Ax\rangle_H^2 \leq \|x\|_H^2\|Ax\|_H^2 = \langle Ax, Ax\rangle_H$$

Therefore,

$$\mathbb{R} \ni \varsigma(A,x) := \sqrt{\langle Ax, Ax\rangle_H - \langle x, Ax\rangle_H^2}.$$

For any $\lambda, \mu \in \mathbb{R}$, the commutators $[A, B]$ and $[A - \lambda, B - \mu]$ agree:

$$\begin{aligned}[A-\lambda, B-\mu] &= (A-\lambda)(B-\mu) - (B-\mu)(A-\lambda) \\ &= AB - \mu A - \lambda B + \lambda\mu - BA + \lambda B + \mu A - \lambda\mu = [A, B].\end{aligned}$$

Since A is symmetric and $\lambda \in \mathbb{R}$, the operator $\tilde{A} = A - \lambda$ is also symmetric on $D_{\tilde{A}} = D_A$. Moreover, for any $x \in D_A$,

$$\begin{aligned}\|\tilde{A}x\|_H^2 &= \langle \tilde{A}x, \tilde{A}x\rangle_H = \langle Ax - \lambda x, Ax - \lambda x\rangle_H \\ &= \langle Ax, Ax\rangle_H - \lambda\langle x, Ax\rangle_H - \lambda\langle Ax, x\rangle_H + \lambda^2\langle x, x\rangle_H \\ &= \langle Ax, Ax\rangle_H - 2\lambda\langle x, Ax\rangle_H + \lambda^2\langle x, x\rangle_H.\end{aligned}$$

We observe that if we choose $\lambda = \langle x, Ax\rangle_H \in \mathbb{R}$ and if $\|x\|_H = 1$, then

$$\|\tilde{A}x\|_H^2 = \langle Ax, Ax\rangle_H - \langle x, Ax\rangle_H^2 = \varsigma(A, x)^2.$$

Now, let $x \in D_{[A,B]} := D_A \cap D_B$ with $\|x\|_H = 1$ be arbitrary. Since the operators $\tilde{A} := A - \langle x, Ax\rangle_H$ and $\tilde{B} := B - \langle x, Bx\rangle_H$ are symmetric, part (i) applies and yields

$$\Big|\langle x, [A, B]x\rangle_H\Big| = \Big|\langle x, [\tilde{A}, \tilde{B}]x\rangle_H\Big| \leq 2\|\tilde{A}x\|_H\|\tilde{B}x\|_H = 2\varsigma(A, x)\,\varsigma(B, x).$$

(iii) Suppose, $B\colon H \to H$ with finite operator norm and $A\colon D_A \subset H \to H$ satisfy

$$[A, B] = i\,\mathrm{id}_{D_{[A,B]}}.$$

By assumption, $D_{[A,B]} = D_A \cap H = D_A$ and $B(D_A) \subset D_A$. In particular, for any $n \in \mathbb{N}$ the inclusion $B^n(D_A) \subset D_A$ is satisfied, which is necessary to define $[A, B^n]$. We prove $[A, B^n] = niB^{n-1}$ by induction. For $n = 1$, the claim holds by assumption. Suppose, it is true for some $n \in \mathbb{N}$. Then

$$\begin{aligned}[A, B^{n+1}] &= AB^{n+1} - B^{n+1}A = \Big(AB^n - B^nA + B^nA\Big)B - B^{n+1}A \\ &= \Big([A, B^n] + B^nA\Big)B - B^{n+1}A = niB^{n-1}B + B^nAB - B^{n+1}A \\ &= niB^n + B^n[A, B] = niB^n + iB^n = (n+1)iB^n.\end{aligned}$$

A consequence is that B cannot be nilpotent: if $B^n = 0$ for some $n \in \mathbb{N}$, then $B^{n-1} = \frac{1}{ni}[A, B^n] = 0$ which iterates to $B = 0$ in contradiction to $[A, B] \neq 0$. Suppose, A has finite operator norm $\|A\|$. Then,

$$n\|B^{n-1}\| = \|[A, B^n]\| \leq \|AB^n\| + \|B^nA\| \leq 2\|A\|\|B^{n-1}\|\|B\|.$$

Since $\|B^{n-1}\| \neq 0$, we obtain $2\|A\| \geq \frac{n}{\|B\|}$ which contradicts $n \in \mathbb{N}$ being arbitrary.

(iv) If $f \in C^1([0,1];\mathbb{C})$, then f' is bounded and in particular $f' \in L^2([0,1];\mathbb{C})$. The map $[0,1] \ni s \mapsto s$ is also bounded. Therefore, the linear operators

$$\begin{aligned}P\colon C^1_0([0,1];\mathbb{C}) &\to L^2([0,1];\mathbb{C}), \qquad & Q\colon L^2([0,1];\mathbb{C}) &\to L^2([0,1];\mathbb{C}) \\ f(s) &\mapsto if'(s) & f(s) &\mapsto sf(s)\end{aligned}$$

are indeed well-defined. They are also symmetric. For Q this follows trivially from $s \in [0,1] \subset \mathbb{R}$. Given any $f, g \in D_P := C^1_0([0,1];\mathbb{C})$, we have

$$\langle Pf, g\rangle_{L^2} = \int_0^1 if'(s)\overline{g}(s)\,\mathrm{d}s = -\int_0^1 if(s)\overline{g}'(s)\,\mathrm{d}s = \int_0^1 f(s)\overline{ig'(s)}\,\mathrm{d}s = \langle f, Pg\rangle_{L^2}.$$

When integrating by parts, the boundary terms vanish due to $f(0) = 0 = f(1)$. Hence, $P\colon C^1_0([0,1];\mathbb{C}) \to L^2([0,1];\mathbb{C})$ is symmetric (but *not* self-adjoint! see Beispiel 6.6.1).

Next, we verify that the commutator $[P,Q]$ is well-defined. Since $D_Q = L^2([0,1];\mathbb{C})$ is the whole space, the only thing to check is that $Qf\colon s \mapsto sf(s)$ is in $D_P = C^1_0([0,1];\mathbb{C})$ whenever $f \in D_{[P,Q]} = C^1_0([0,1];\mathbb{C})$. But this follows trivially from the product rule. Moreover,

$$([P,Q]f)(s) = (P(Qf))(s) - (Q(Pf))(s) = if(s) + isf'(s) - sif'(s) = if(s)$$

for almost every $s \in [0,1]$ which proves that (P,Q) is a Heisenberg pair. Finally, by part (ii), we have

$$\forall f \in C^1_0,\ \|f\|_{L^2} = 1: \quad \varsigma(P,f)\,\varsigma(Q,f) \geq \tfrac{1}{2}\Big|\langle f, [P,Q]f\rangle_{L^2}\Big| = \tfrac{1}{2}\Big|\langle f, if\rangle_{L^2}\Big| = \tfrac{1}{2}.$$