ANY ALGEBRAIC BASIS IS UNCONTABLE

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Definition 0.1. *Let X be a vector space over a field $\mathbb{K}$. An* algebraic basis *for X is a subset $E \subset X$ such that every $x \in X$ is uniquely given as* finite *linear combination of elements in E (with coefficients in $\mathbb{K}$).*

The aim of this short note is to give a proof of the following basic assertion:

Theorem 0.2. *Let $(X, \|\cdot\|)$ be a complete normed space, i.e. a Banach space. Any countable algebraic basis for X is actually finite (hence X has finite dimension).*

Remark 0.3. *Here and throughout the course it is assumed (as it is customary) that normed vector spaces are either vector spaces over $\mathbb{R}$ or $\mathbb{C}$. This allows, in particular, to 'rescale vectors by their norm'.*

Remark 0.4. *We tacitly give for granted the (non-trivial) fact that any vector space over a field $\mathbb{K}$ admits an algebraic basis: this follows via an argument that is similar in spirit to the one you have seen in earlier courses for the finite-dimensional case, but does require Zorn's Lemma (or, equivalently, the axiom of choice). The moral of the previous theorem is that any infinite-dimensional Banach space only admits* uncountable *algebraic bases. This is one of the reasons (yet not the only one) why the notion of algebraic basis is not really that useful in Functional Analysis.*

Proof. Let E be a countable basis of X, which we can enumerate as $\{e_1, e_2, \ldots\}$. (Of course, we convene that, if the basis is finite, then we stop enumerating after finitely many steps). For $n \in \mathbb{N}_* = \mathbb{N} \setminus \{0\}$ we define the linear subspaces $A_n = \mathrm{span}\{e_1, \ldots, e_n\} \subset X$.

First of all, observe that by the assumption that E is an algebraic basis of X we trivially have that

$$X = \bigcup_{n \in \mathbb{N}_*} A_n.$$

Then we proceed with two claims:

Claim 1: as finite dimensional subspace, A_n is closed.

Indeed, we proved that any two norms on a finite-dimensional vector space are always equivalent, so in particular we can consider on A_n the norm $\|\cdot\|$ obtained by restriction of the ambient norm on X, and the Euclidean norm defined by letting

$$\|\sum_{i=1}^{n} \lambda_i e_i\|' = \sqrt{\sum_{i=1}^{n} |\lambda_i|^2}.$$

We then showed (cf. e.g. Exercise 1.1) that there exists a positive constant $C > 0$ such that

$$C^{-1}\|x\|' \leq \|x\| \leq C\|x\|', \ \forall x \in A_n.$$

This implies at once that a sequence is Cauchy (respectively: convergent) in $(A_n, \|\cdot\|)$ if and only if it is Cauchy (respectively: convergent) in $(A_n, \|\cdot\|')$.

But, by completeness of Euclidean $\mathbb{R}^n$ we know that $(A_n, \|\cdot\|')$ is complete, and so (by virtue of the previous remark) we conclude that $(A_n, \|\cdot\|)$ is also complete. Lastly, recall that a subspace of a complete metric space is complete if and only if it is closed, so $A_n \subset X$ is closed, like we had claimed.

Claim 2: As a subspace of X, we have that A_n has empty interior unless X is finite-dimensional.

Suppose, on the contrary, that there exist $x \in A_n$ and $\varepsilon > 0$ such that $B_\varepsilon(x) \subset A_n$. Since A_n is a linear subspace, we may subtract $x \in A_n$ from the elements in $B_\varepsilon(x)$ to obtain $B_\varepsilon(0) \subset A_n$. For the same reason,

$$A_n \supset \{\lambda y \ : \ \lambda > 0, \ y \in B_\varepsilon(x)\} = X.$$

This however implies $\dim X \leq n$ which in turn forces X to have finite dimension. Thus, if X is infinite-dimensional then $(\overline{A_n})^\circ = A_n^\circ = \emptyset$ which means that A_n is nowhere dense.

If we now combine together the two claims above and Baire's theorem we immediately reach a contradiction unless the vector space X has finite dimension, which is precisely what we had to prove.

□

Remark 0.5. *Let X be the space of polynomials $p\colon [0,1] \to \mathbb{R}$ with real coefficients endowed with the norm $\|\cdot\|_{C^0([0,1])}$. Let $f_n\colon [0,1] \to \mathbb{R}$ be given by the monomial $f_n(x) = x^n$. Then, $\{f_n \ : \ n \in \mathbb{N}\}$ is a countable algebraic basis for X. Hence, according to the theorem above, the space $(X, \|\cdot\|_{C^0([0,1])})$ must be incomplete. Bonus question: can you characterise the metric completion of X? (Equivalently: can you determine the closure of X inside $C^0([0,1])$?)*

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