SPECTRAL THEORY OVER THE REAL FIELD, A VERY SHORT NOTE

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This informal note briefly discusses the problem of defining the spectrum of a linear operator acting on a real Banach space, and the construction of a spectral theory over $\mathbb{R}$ given the (well-known) spectral theory over $\mathbb{C}$.

1. The conceptual path

Let $(X, \|\cdot\|_X)$ be a Banach space over the real field $\mathbb{R}$ and let $A : D_A \subset X \to X$ be $\mathbb{R}$-linear, where D_A is as usual a real subspace, which we shall assume to be dense in X. We would like to define the spectrum of A and see to what extent the theory we have developed over the complex field $\mathbb{C}$ can be recovered in this case (which is very important in several applications).

At the abstract level, one can take two different approaches.

Approach 1: complexification.

Given X as above, we define its complexification as the tensor product $X_{\mathbb{C}} := X \otimes_{\mathbb{R}} \mathbb{C}$ (where $\mathbb{C}$ is regarded as a two-dimensional vector space over $\mathbb{R}$, in the standard sense). There is then a standard identification: $X_{\mathbb{C}} \simeq X^2$ where the action of complex scalars on $\mathbb{C}$ is defined as follows:

$$(\alpha + i\beta)(x_1, x_2) = (\alpha x_1 - \beta x_2, \beta x_1 + \alpha x_2).$$

In practice the couple (x_1, x_2) stands for the complex vector $x_1 + ix_2$ and the operation above is defined consistently with this idea.

Remark 1.1. *In case X is a space of $\mathbb{R}$-valued functions, then $X_{\mathbb{C}}$ has a particularly simple description. For instance $L^p(\Omega; \mathbb{R})_{\mathbb{C}} \equiv L^p(\Omega; \mathbb{C})$, and analogously for space of continuous, differentiable functions etc...*

At that stage, one considers the complexification of the subspace D_A (denoted by $(D_A)_{\mathbb{C}}$) and the complexified extension of A (denoted by $A_{\mathbb{C}}$) defined by *componentwise action* i. e.

$$A_{\mathbb{C}} : (D_A)_{\mathbb{C}} \subset X_{\mathbb{C}} \to X_{\mathbb{C}}, \quad A_{\mathbb{C}}(x_1, x_2) = (Ax_1, Ax_2).$$

It is readily checked that $A_{\mathbb{C}}$ is densely defined and $\mathbb{C}$-linear. Hence, we can define its spectrum $\sigma(A_{\mathbb{C}})$, its resolvent $\rho(A_{\mathbb{C}})$ and the three parts of the spectrum $\sigma_p(A_{\mathbb{C}}), \sigma_c(A_{\mathbb{C}}), \sigma_r(A_{\mathbb{C}})$. At that stage, one can then consider the real part of the spectrum $(\sigma(A_{\mathbb{C}}) \cap \mathbb{R})$, of the resolvent $\rho(A_{\mathbb{C}}) \cap \mathbb{R}$ and of the components of the spectrum $\sigma_p(A_{\mathbb{C}}) \cap \mathbb{R}, \sigma_c(A_{\mathbb{C}}) \cap \mathbb{R}, \sigma_r(A_{\mathbb{C}}) \cap \mathbb{R}$.

This approach is conceptually very neat and allows exploiting all tools we have developed over the algebraically closed field $\mathbb{C}$ (for instance: the fundamental characterization of the spectral radius $r_{A_{\mathbb{C}}} = \lim_{n\to\infty} \|A^n_{\mathbb{C}}\|^{1/n}$), but *a priori* has the disadvantage of giving some unnatural definitions. For instance, notice that

$$\sigma_p(A_{\mathbb{C}}) \cap \mathbb{R} := \{\lambda \in \mathbb{R} \ : \ (\lambda - A_{\mathbb{C}}) : (D_A)_{\mathbb{C}} \subset X_{\mathbb{C}} \to X_{\mathbb{C}} \text{ is not injective}\}$$

so a priori we are considering here *complex eigenvectors/eigenspaces*, which are not really objects that are associated to A, but solely to $A_{\mathbb{C}}$.

Approach 2: purely real theory.

Alternatively, one can take a totally different approach and formally recycle over $\mathbb{R}$ the *definitions* given in the complex case. So, one shall define:

Definition 1.2. *Let $(X, \|\cdot\|_X)$ be a Banach space over the real field $\mathbb{R}$ and let $A : D_A \subset X \to X$ be a densely defined $\mathbb{R}$-linear map. Then we define the real resolvent of A as*

$$\rho^{(\mathbb{R})}(A) := \left\{\lambda \in \mathbb{R} \ : \ (\lambda - A) : D_A \subset X \to X \ \textit{bijective}, \ \exists \ (\lambda - A)^{-1} \in L(X)\right\}$$

and the real spectrum of A as

$$\sigma^{(\mathbb{R})}(A) := \mathbb{R} \setminus \rho^{(\mathbb{R})}(A).$$

The spectrum is partitioned as follows:

$$\sigma_p^{(\mathbb{R})}(A) := \{\lambda \in \mathbb{R} \ : \ (\lambda - A) : D_A \subset X \to X \ \textit{not injective}\}\,;$$

$$\sigma_c^{(\mathbb{R})}(A) := \{\lambda \in \mathbb{R} \ : \ (\lambda - A) : D_A \subset X \to X \ \textit{injective, with dense range}\}\,;$$

and

$$\sigma_r^{(\mathbb{R})}(A) := \sigma^{(\mathbb{R})}(A) \setminus (\sigma_p^{(\mathbb{R})}(A) \cup \sigma_c^{(\mathbb{R})}(A)).$$

Now, observe that the elements of $\sigma_p^{(\mathbb{R})}(A)$ can be legitimately called eigenvalues of A. This theory is the natural analogue of the spectral theory for finite-dimensional vector spaces over $\mathbb{R}$.

The connection between the two perspectives is given by the following theorem, whose (very simple!) proof is left as an exercise.

Theorem 1.3. *Let $(X, \|\cdot\|_X)$ be a Banach space over the real field $\mathbb{R}$ and let $A : D_A \subset X \to X$ be a densely defined $\mathbb{R}$-linear map. Then the following equalities hold true:*

$$\sigma^{(\mathbb{R})}(A) = \sigma(A_{\mathbb{C}}) \cap \mathbb{R}, \ \ \rho^{(\mathbb{R})}(A) = \rho(A_{\mathbb{C}}) \cap \mathbb{R}$$

as well as

$$\begin{cases} \sigma_p^{(\mathbb{R})}(A) := \sigma_p(A_{\mathbb{C}}) \cap \mathbb{R} \\ \sigma_c^{(\mathbb{R})}(A) := \sigma_c(A_{\mathbb{C}}) \cap \mathbb{R} \\ \sigma_r^{(\mathbb{R})}(A) := \sigma_r(A_{\mathbb{C}}) \cap \mathbb{R}. \end{cases}$$

This fact allows to connect the two perspectives, and exploit in the real case all the tools that have been developed in the complex context. In particular, let us explicitly state the spectral theorem for bounded operators.

Theorem 1.4. *Let $(H, (\cdot, \cdot)_H)$ be an infinite-dimensional Hilbert space over the real field $\mathbb{R}$ and let $T \neq 0 \in L(H)$ be symmetric (equivalently: self-adjoint) and compact. Then there exists a sequence (λ_k) of real eigenvalues* (possibly with repetitions) *such that $\lambda_k \to 0$ as one lets $k \to \infty$ and an associated orthonormal sequence of eigenfunctions (e_k) with*

$$H = \ker(T) \oplus \overline{span_{\mathbb{R}} \{e_k \ : \ k \in \mathbb{N}\}}$$

and for any $x \in H$

$$Tx = \sum_{k \geq 0} \lambda_k (x, e_k)_H \ e_k.$$

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