HILBERTIAN BASES AND APPLICATIONS

ALESSANDRO CARLOTTO

This informal note presents the notion of Hilbertian basis and discusses some related topics, namely Bessel's inequality, Parseval's identity and the classification problem for real Hilbert spaces.

1. The setup

Throughout this note, we let $(H, \langle , \rangle)$ denote a Hilbert space over $\mathbb{R}$. The results we are about to present can easily be extended to the case when the base field is $\mathbb{C}$, with changes of purely notational character. To avoid annoying (albeit trivial) subcases, we further convene that H has infinite dimension. We shall start by recalling the following fundamental result about the existence of a continuous projection operator onto any given closed subspace $K \subset H$:

Theorem 1.1. *Let $(H, \langle , \rangle)$ be a Hilbert space and let $K \subset H$ be a closed linear subspace. Then K has an orthogonal topological complement, namely*

$$H = K \oplus^{\perp} K^{\perp}$$

and there exist continuous, linear operators $\pi_K, \pi_{K^\perp} \in L(H)$ such that the following assertions hold true:

$$\begin{aligned} &\|\pi_K\|_{L(H)} = 1, && \|\pi_{K^\perp}\|_{L(H)} = 1; \\ &[\pi_K]_{|K} = id_K, && [\pi_{K^\perp}]_{|K^\perp} = id_{K^\perp}; \\ &\pi_K^2 = \pi_K, && \pi_{K^\perp}^2 = \pi_{K^\perp}; \\ &id - \pi_K = \pi_{K^\perp}, && id - \pi_{K^\perp} = \pi_K. \end{aligned}$$

Here $id : H \to H$ denotes the identity map of H, ad similarly id_K (resp. $id_{K^\perp}$) its restriction to K (resp. $K^\perp$).

We also need to recall the variational characterization of the projections:

Theorem 1.2. *Let $(H, \langle , \rangle)$ be a Hilbert space and let $K \subset H$ be a closed linear subspace. Given $x \in H$, the following two assertions are equivalent for a vector $y \in K$:*

i) $d(x, y) = d(x, K)$ i. e. y realizes the distance of x from the subspace K;
ii) $\pi_K(x) = y$ i. e. y is the projection of x on the subspace K.

As a simple application, we can consider the important case when the subspace in question is finite-dimensional.

Corollary 1.3. *Let* $(H, \langle , \rangle)$ *be a Hilbert space, let* $H_N \subset H$ *be a subspace such that* $dim_{\mathbb{R}}(H_N) = N$ *and consider an orthonormal basis thereof* $\{e_1, \ldots, e_N\}$*. Then for any* $x \in H$ *we have*

$$\pi_{H_N}(x) = \sum_{k=1}^{N} \langle x, e_k \rangle e_k.$$

The proof of such assertion is straightforward and relies on checking that indeed the vector

$$x - \sum_{k=1}^{N} \langle x, e_k \rangle e_k$$

is orthogonal to e_i for each $i = 1, \ldots, N$ hence to H_N.

We give the following preliminary definition:

Definition 1.4. *Let* $(H, \langle , \rangle)$ *be a Hilbert space. For a set* I*, we shall say that* $(e_i)_{i \in I}$ *is an orthonormal family if*

$$\langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

for any choice of $i, j \in I$*. If* $I = \mathbb{N}$ *we shall call* $(e_k)_{k \in \mathbb{N}}$ *orthonormal system.*

In general, working with respect to an orthonormal system is very useful whenever dealing with Hilbert spaces. This fact is to a significant extent related to the following theorem.

Theorem 1.5. *Let* $(H, \langle , \rangle)$ *be a Hilbert space and let* $(e_k)_{k \in \mathbb{N}}$ *be an orthonormal system thereof. Then:*

a) for any $x \in H$ *one has*

$$\sum_{k=0}^{\infty} |\langle x, e_k \rangle|^2 \leq \|x\|^2; \tag{1.1}$$

b) for any $x \in H$ *the series*

$$\sum_{k=0}^{\infty} \langle x, e_k \rangle e_k \tag{1.2}$$

converges;

c) given $x \in H$ *the equality*

$$\sum_{k=0}^{\infty} |\langle x, e_k \rangle|^2 = \|x\|^2$$

holds if and only if

$$x = \sum_{k=0}^{\infty} \langle x, e_k \rangle e_k.$$

The series given in equation (1.2) is called *Fourier series* of $x \in H$, the inequality (1.1) *Bessel's inequality* and when equality holds we call it *Parseval identity* instead.

Proof. For part a) set $D_N := \operatorname{span}_{\mathbb{R}}\{e_0, \dots, e_N\}$, and observe that

$$\|x\|^2 = \|\pi_{D_N}(x)\|^2 + \|\pi_{D_N^\perp}\|^2$$

(which follows from Theorem 1.1) trivially implies

$$\|x\|^2 \geq \|\pi_{D_N}(x)\|^2 = \sum_{k=0}^{N} |\langle x, e_k\rangle|^2$$

via Corollary 1.3. But now such uniform bound holds true for any $N \in \mathbb{N}$, thus the conclusion.

For part b), let $S_n := \sum_{k=0}^{n}\langle x, e_k\rangle e_k$ thus for $l_2 \geq l_1$ one has

$$\|S_{l_2} - S_{l_1}\|^2 = \|\sum_{k=l_1+1}^{l_2} \langle x, e_k\rangle e_k\|^2 = \sum_{k=l_1+1}^{l_2} |\langle x, e_k\rangle|^2$$

which implies that the sequence of partial sums is Cauchy in H, thus convergent.

For part c), it is sufficient to consider the identity

$$\Big\|x - \sum_{k=0}^{N}\langle x, e_k\rangle e_k\Big\|^2 = \|x\|^2 - \sum_{k=0}^{N} |\langle x, e_k\rangle|^2$$

which comes just by expanding the square, and let $N \to \infty$. □

Based on the above discussion, we give the following:

Definition 1.6. *Let $(H, \langle,\rangle)$ be a Hilbert space. We say that an orthonormal system $(e_k)_{k\in\mathbb{N}}$ is an Hilbertian basis if*

$$x = \sum_{k=0}^{\infty}\langle x, e_k\rangle e_k \quad \forall x \in H.$$

Remark 1.7. *Consider the space ℓ^2 of sequences having summable squares. Then we claim that the orthonormal family given by the monomial sequences*

$$e_i = (0, \dots, 0, 1, 0, \dots)$$

(so having 1 only in the i-th slot) is indeed an Hilbertian basis. To this scope, based on the criterion provided above (part c)) it is enough to observe that for any $x := (x_k)_{k\in\mathbb{N}} \in \ell^2$ one has that

$$\|x\|^2_{\ell^2} = \sum_{k=0}^{\infty} x_k^2 = \sum_{k=0}^{\infty} |\langle x, e_k\rangle|^2$$

where the first equality is the very definition of ℓ^2-norm, and the second relies on the definition of the basis we are working with. As we will see below ℓ^2 plays the role of canonical model for all separable *Hilbert spaces, in the same way $\mathbb{R}^n$ models any vector space V over $\mathbb{R}$ such that $\dim_{\mathbb{R}}(V) = n$.*

2. Separability

The following proposition provides a simple criterion to determine whether a given Hilbert space H does admit a complete orthonormal system, namely an Hilbertian basis.

Proposition 2.1. *A Hilbert space $(H, \langle , \rangle)$ admits an Hilbertian basis if and only if its is separable.*

Proof. Assume first $(H, \langle , \rangle)$ admits an Hilbertian basis $(e_k)_{k\in\mathbb{N}}$. Then the countable subset D consisting of finite linear combinations, with coefficients in $\mathbb{Q}$, of elements belonging to such basis is dense in H. Indeed, given $x \in H$ and $\varepsilon > 0$ one can find (thanks to part c) of Theorem 1.5) an integer $N = N(\varepsilon)$ such that

$$d_H^2\big(x, \sum_{k=0}^{N} \langle x, e_k\rangle e_k\big) = \sum_{k=N+1}^{\infty} |\langle x, e_k\rangle|^2 < \frac{\varepsilon}{2}$$

but then we can approximate each coefficient $\langle x, e_k\rangle$ by means of $q_k \in \mathbb{Q}$ in a way that

$$d_H^2(x, q_k e_k) < \varepsilon$$

which implies our claim.

Conversely, let $(x_k)_{k\in\mathbb{N}}$ an enumeration of a countable dense subset of H. Before proceeding further, we need to make two preliminary operations. First, we can set $v_0 = x_0$ and then, given $\{v_0, v_1, \dots, v_n\}$ and proceeding inductively, we let $v_{n+1} = x_{k(n+1)}$ where the positive integer $k(n+1)$ in the sequence $n \mapsto k(n)$ is defined by the requirement that

$$k(n+1) = \min\{p \in \mathbb{N} \ : \ p > k(n), \ x_p \notin \operatorname{span}_{\mathbb{R}}\{v_0, v_1, \dots, v_n\}\ \}.$$

Notice that, if we let D_N to be the linear span of $\{v_0, v_1, \dots, v_N\}$ and $D_\infty = \cup_N D_N$ we have that D_∞ is dense in H because (by construction) $x_k \in D_\infty$ for all $k \in \mathbb{N}$. As a second step, we apply the Gram-Schmidt procedure to the sequence $(v_k)_{k\in\mathbb{N}}$ thereby obtaining an orthonormal system $(e_k)_{k\in\mathbb{N}}$: we claim that such system is in fact an Hilbertian basis for H. Given any $x \in H$ and $N \in \mathbb{N}$, we let

$$d_N = d(x, D_N) = \inf_{y\in D_N} \|x - y\|$$

so that by the density of $D_\infty \subset H$ we get at once that $d_N \downarrow 0$ as one lets $N \to \infty$. But then, given $\varepsilon > 0$ we can find $N = N(\varepsilon)$ such that $d_N < \varepsilon$ and then by Corollary 1.3 this precisely means that

$$\|x - \sum_{k=0}^{N} \langle x, e_k\rangle e_k\| < \varepsilon.$$

By the arbitrariness of $\varepsilon > 0$ we conclude that $x = \sum_{k=0}^{\infty} \langle x, e_k\rangle e_k$, which completes the proof. □

Corollary 2.2. *The space $L^2((-\pi,\pi);\mathbb{R})$ (and, similarly, the space $L^2((-\pi,\pi);\mathbb{C})$) admits an Hilbertian basis.*

The problem, in applying the previous proposition, is that it does not really give a practical criterion to verify whether a given orthonormal system is actually an Hilbertian basis. This is an important problem in Real Analysis. We mention here a fundamental result, that is frequently applied:

Theorem 2.3. *a. An Hilbertian basis for $L^2((-\pi,\pi);\mathbb{R})$ is given by the standard (real) trigonometric system*

$$\frac{1}{\sqrt{2\pi}};\quad \frac{1}{\sqrt{\pi}}\cos(kx)\quad k\in\mathbb{N}_*;\quad \frac{1}{\sqrt{\pi}}\sin(kx)\quad k\in\mathbb{N}_*.$$

b. An Hilbertian basis for $L^2((-,\pi,\pi);\mathbb{C})$ is given by the standard (complex) trigonometric system

$$\frac{1}{\sqrt{2\pi}}e^{ikx}\ \ k\in\mathbb{Z}.$$

For a (streamlined, very readable) proof of the former assertion, the reader may consult e. g. [ADPM], pp. 75-79. The latter assertion then follows easily from the former by considering the cases when $f\in L^2((-,\pi,\pi);\mathbb{C})$ is actually real-valued, and then the case when $if\in L^2((-,\pi,\pi);\mathbb{C})$ is real-valued.

For $L^2((-,\pi,\pi);\mathbb{R})$, if we write (as it is customary) the Fourier series as

$$S(x)=\frac{1}{2}a_0+\sum_{k=1}^{\infty}(a_k\cos(kx)+b_k\sin(kx))$$

for

$$a_k=\frac{1}{\pi}\int_{-\pi}^{\pi}f(y)\cos(ky)\,dy\ \ k\in\mathbb{N},\quad b_k=\frac{1}{\pi}\int_{-\pi}^{\pi}f(y)\cos(ky)\,dy\ \ k\in\mathbb{N}_*$$

then the Parseval identity reads

$$\frac{1}{\pi}\int_{-\pi}^{\pi}|f(x)|^2\,dx=\frac{1}{2}a_0^2+\sum_{k=1}^{\infty}(a_k^2+b_k^2).$$

As an interesting special case, considering the Fourier series of the function $f(x)=x$ one can prove the well-known identity

$$\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}.$$

Similarly, in the complex case, if we write the Fourier series as

$$S_{\mathbb{C}}(x)=\sum_{k\in\mathbb{Z}}c_ke^{ikx}$$

for

$$c_k=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(y)e^{-iky}\,dy$$

then the Parseval identity reads

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2\,dx=\sum_{k\in\mathbb{Z}}|c_k|^2.$$

Remark 2.4. *It is important not to confuse the notion of Hilbertian basis with that of algebraic basis, which has rather limited relevance and utility in the analytical study of infinite-dimensional vector spaces. To keep the two notions well-distinct, we remind the reader that given a real vector space V it is possible to prove (which we did, using Baire's Lemma) that an algebraic basis for V has never countable cardinality (i.e. it has either finite cardinality* *or at least as large as that of* $\mathbb{R}$*), while on the other hand for all separable Hilbert spaces an Hilbertian basis is always countable.*

Remark 2.5. *It is obviously an interesting question whether there exist Hilbert spaces that are not separable. The simplest such example, yet already a nontrivial one, can be constructed as follows. Let* H *be the set of functions* $f : [0,1] \to \mathbb{R}$ *such that* $f(x) \neq 0$ *for at most countably many* $x \in [0,1]$ *and such that* $\sum_{x\in[0,1]} |f(x)|^2 < +\infty$*. Endow this set with the scalar product over* $\mathbb{R}$ *given by*

$$\langle f, g\rangle = \sum_{x\in[0,1]} f(x)g(x).$$

It is possible to show that this couple $(H, \langle,\rangle)$ *is indeed a non-separable Hilbert space.*

3. Isomorphic classification

Based on the remark above, we consider the problem of classifying Hilbert spaces up to isometric isomorphism. Precisely, we mean the following.

Definition 3.1. *Given Hilbert spaces* $(H_1, \langle,\rangle_1)$ *and* $(H_2, \langle,\rangle_2)$ *we shall say that a linear map* $\Psi : H_1 \to H_2$ *is an isometric isomorphism if it is a bijection and for any* $x_1, z_1 \in H_1$ *one has*

$$\langle x_1, z_1\rangle_1 = \langle \Psi(x_1), \Psi(z_1)\rangle_2.$$

We say that $(H_1, \langle,\rangle_1)$ *and* $(H_2, \langle,\rangle_2)$ *are equivalent, and write* $(H_1, \langle,\rangle_1) \simeq (H_2, \langle,\rangle_2)$ *if there is an isometric isomorphism from the former space to the latter.*

The previous definition is indeed well-posed, for it is easy to check that $\simeq$ is an equivalence relation.

We start by observing that, like in the finite-dimensional context, the cardinality of an Hilbertian basis (that is to say, generalizing the one above: of an orthonormal family, whose finite linear combinations are dense in the whole space in question) is indeed an invariant.

Definition 3.2. *Let* $(H, \langle,\rangle)$ *be a Hilbert space. We say that an orthonormal family* $(e_i)_{i\in I}$ *is a generalized Hilbertian basis for* $(H, \langle,\rangle)$ *if*

$$\overline{\left\{ \sum_{J\subset I, J finite} x_i e_i, \ x_i \in \mathbb{R} \right\}}^H = H.$$

The following three theorems are stated here without proof, for which we refer the reader for instance to [KG] part III, section 4.1.

Theorem 3.3. *Any Hilbert space* $(H, \langle,\rangle)$ *admits a generalized Hilbertian basis* $\mathscr{B}$*.*

The statement above is in fact a rather standard application of Zorn's Lemma.

Theorem 3.4. *Let* $(H, \langle,\rangle)$ *be a Hilbert space and let* $\mathscr{B}, \mathscr{B}'$ *be two generalized Hilbertian bases for* $(H, \langle,\rangle)$*. Then there is a bijection* $\Omega : \mathscr{B} \to \mathscr{B}'$*.*

Definition 3.5. *Let* $(H, \langle,\rangle)$ *be a Hilbert space. The cardinality of a basis (hence of any basis) of* $(H, \langle,\rangle)$ *is called Hilbertian dimension of* $(H, \langle,\rangle)$*.*

That being said, one can show that the Hilbertian dimension (i. e. the cardinality of a generalized basis) is indeed the only invariant that comes into play in the classification of Hilbert spaces up to isometric isomorphism, as encoded in the following assertion.

Theorem 3.6. *Two Hilbert spaces $(H_1, \langle , \rangle_1)$ and $(H_2, \langle , \rangle_2)$ are isometrically isomorphic if and only if there exist generalized Hilbertian bases $\mathscr{B}_1$ of $(H_1, \langle , \rangle_1)$ and, respectively, $\mathscr{B}_2$ of $(H_2, \langle , \rangle_2)$, having the same cardinality.*

As a special case, this implies the aforementioned classification result:

Corollary 3.7. *Any two separable Hilbert spaces are isometrically isomorphic, and each of them is isometrically isomorphic to ℓ^2.*

References

[ADPM] L. Ambrosio, G. Da Prato, A. Mennucci, *Introduction to measure theory and integration.* Edizioni della Normale, Pisa, 2011, xii+187 pp.

[KG] A. A. Kirillov, A. D. Gvishiani, *Theorems and problems in functional analysis.* Springer-Verlag, New York-Berlin, 1982, ix+347 pp.

ETH - Department of Mathematics, ETH, Zürich, Switzerland
Email address: alessandro.carlotto@math.ethz.ch
URL: https://people.math.ethz.ch/~ac/