EQUIVALENT NOTIONS OF COMPACTNESS

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Lemma 1. *Let (X, d) be a metric space and assume that X is separable, i.e., it contains a countable dense subset. Prove that any open cover $\mathcal{O}$ of X admits a countable subcover.*

Proof. First observe that X admits a countable basis for the topology generated by the metric d. Indeed, denoting by D a countable dense subset of X, the countable set

$$\mathcal{B} := \{B(x, q) \,:\, x \in D, q \in \mathbb{Q}_{>0}\}$$

is a basis for the topology of metric space on X. Indeed consider any open set $U \subseteq X$ and any point $x_0 \in U$. By definition of metric topology, there exists $y \in U$ and $r > 0$ such that $x_0 \in B(y, r) \subseteq U$. Then observe that $B(x_0, r') \subseteq B(y, r) \subseteq U$ with $r' := r - d(x_0, y) > 0$, by the triangle inequality. Since D is dense, there exists $x \in D \cap B(x_0, r'/2)$, therefore we have that $x_0 \in B(x, r'/2) \subseteq B(x_0, r') \subseteq U$, again by triangle inequality. Now take $q \in \mathbb{Q}_{>0}$ such that $d(x_0, x) < q < r'/2$ (which is possible since $d(x_0, x) < r'/2$), then $x_0 \in B(x, q) \subseteq B(x, r'/2) \subseteq U$. Note that $B(x, q) \in \mathcal{B}$, thus we proved that $\mathcal{B}$ is a basis for the topology.

Note. Here we proved that any separable metric space is second-countable. However, this is not true in general. In fact there exist separable first-countable topological spaces that are not second-countable.

Now we want to prove that, if a topological space X has a countable basis for its topology, then every open cover admits a countable subcover. Denote by $\mathcal{B} = \{B_n\}_{n\in\mathbb{N}}$ a countable basis of the topology and consider an open cover $\mathcal{O}$. Define $I \subset \mathbb{N}$ as the set of indices $n \in \mathbb{N}$ such that there exists $O_n \in \mathcal{O}$ containing B_n. Then define $\mathcal{O}' := \{O_n \,:\, n \in I\}$, where for every $n \in I$ we make a choice of $O_n \in \mathcal{O}$ such that $B_n \subseteq O_n$. We claim that $\mathcal{O}'$ is a countable subcover of $\mathcal{O}$. The fact that $\mathcal{O}'$ is countable is obvious, hence let us prove that it is a cover. Consider $x \in X$, then there exists $O \in \mathcal{O}$ such that $x \in O$. Since $\mathcal{B}$ is a basis for the topology, we can pick $B_n \in \mathcal{B}$ such that $x \in B_n \subseteq O$. In particular $n \in I$, hence $x \in B_n \subseteq O_n$ for some $O_n \in \mathcal{O}'$, which proves that $\mathcal{O}'$ is a cover. □

Proposition 2. *Given a metric space (X, d), the following conditions are equivalent:*

- (C1) *The space X is compact (i.e., every open cover of X admits a finite subcover).*
- (C2) *The space X is sequentially compact (i.e., every sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ admits a converging subsequence).*
- (C3) *The space X is complete (i.e., every Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to some $x \in X$) and totally bounded (i.e., for every $\varepsilon > 0$ there exists a finite set of points $x_1, \dots, x_k \in X$ such that $X \subseteq \cup_{i=1}^k B(x_i, \varepsilon)$).*

Proof. We prove that (C1) is equivalent to (C2), which is in turn equivalent to (C3).

"(C1) $\Longrightarrow$ (C2)". Assume by contradiction that X is compact but not sequentially compact. In particular there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ without converging subsequences. Then define

$$\mathcal{O} := \{O \subseteq X \,:\, O \text{ open},\ O \text{ contains a finite number of elements in } \{x_n\}_{n\in\mathbb{N}}\}.$$

Observe that $\mathcal{O}$ is an open cover of X. Indeed, for every $x \in X$, there exists an open neighborhood O of x that does not contain elements of $\{x_n\}_{n\in\mathbb{N}}$ eventually in $n \in \mathbb{N}$ (otherwise it would exists a subsequence of $\{x_n\}_{n\in\mathbb{N}}$ converging to x). Hence, by compactness of X, there exists a finite subcover $\mathcal{O}'$ of $\mathcal{O}$. Since $\mathcal{O}'$ is a finite cover of X and $\{x_n\}_{n\in\mathbb{N}}$ is an infinite sequence, there exists $O' \in \mathcal{O}'$ that contains an infinite number of elements of $\{x_n\}_{n\in\mathbb{N}}$. However, this contradicts the fact that $O' \in \mathcal{O}' \subseteq \mathcal{O}$.
"(C2) $\implies$ (C1)". Let us assume that X is sequentially compact, we want to prove that it is compact. We first show that X is separable. First observe that X is bounded, otherwise it is easy to construct a sequence without converging subsequences ("going to infinity"). We construct the following sequence: we fix some $x_0 \in X$ and then we define x_{n+1} for $n \geq 0$ in such a way that

$$\min_{i=1,\dots,n} d(x_{n+1}, x_i) \geq \frac{1}{2} \sup_{x\in X} \min_{i=1,\dots,n} d(x, x_i). \tag{1}$$

Observe that the term on the right hand side is finite by boundedness of X, thus it is possible to find x_{n+1} as required. We want to show that $\{x_n\}_{n\in\mathbb{N}}$ is a dense subset of X. Since X is sequentially compact, $\{x_n\}_{n\in\mathbb{N}}$ admits a converging subsequence $\{x_{n_m}\}_{m\in\mathbb{N}}$. This implies that $\min_{i=1,\dots,n_m-1} d(x_{n_m}, x_i) \leq d(x_{n_m}, x_{n_{m-1}})$ converges to 0 as $m \to \infty$. As a result $\sup_{x\in X} \min_{i=1,\dots,n_m} d(x, x_i) \to 0$ as $m \to \infty$, by (1). However, from this it follows directly that

$$\sup_{x\in X} \min_{i=1,\dots,n} d(x, x_i) \to 0 \quad \text{as } n \to \infty,$$

which proves that $\{x_n\}_{n\in\mathbb{N}}$ is a (countable) dense subset of X.

Now consider an open cover $\mathcal{O}$ of X. Since X is separable, we can extract a countable subcover $\mathcal{O}' = \{O_k\}_{k\in\mathbb{N}}$ of $\mathcal{O}$ by the Lemma above. Assume by contradiction that $\mathcal{O}'$ does not admit any finite subcover, then $\cup_{k=1}^n O_k \neq X$ for any $n \in \mathbb{N}$. In particular there exists $x_n \in X \setminus \cup_{k=1}^n O_k$ for any $n \in \mathbb{N}$. Since X is sequentially compact, the sequence $\{x_n\}_{n\in\mathbb{N}}$ has a subsequence converging to some point $x \in X$. However observe that $x \in X \setminus \cup_{k=1}^n O_k$ for all $n \in \mathbb{N}$, since the sequence $\{x_n\}_{n\in\mathbb{N}}$ is eventually contained in $X \setminus \cup_{k=1}^n O_k$, which is closed. Therefore, $x \in X \setminus \cup_{n\in\mathbb{N}} O_n$, which is a contradiction since $X \setminus \cup_{n\in\mathbb{N}} O_n = \emptyset$.
"(C2) $\implies$ (C3)". Let us assume that X is sequentially compact. Given a Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$, there exists a subsequence converging to some $x \in X$. However, it is not difficult to check (do it!) that if a subsequence of a Cauchy sequence converges to some point $x \in X$, then the whole sequence converges to such a point. Therefore X is complete The proof that X is totally bounded is analogous to the proof X is separable in the previous implication (which is (C2) $\implies$ (C1)), hence we leave it for the reader.
"(C3) $\implies$ (C2)". Let us assume that X is complete and totally bounded and consider a sequence $\{x_n\}_{n\in\mathbb{N}}$. We want to prove that this sequence admits a converging subsequence. Since X is totally bounded, for every $m \in \mathbb{N}$ there exists a finite cover $\mathcal{O}_m$ of metric balls of radius $1/m$. Note that there exists a subsequence $\{x_n^1\}_{n\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that all its elements are contained in the same $O_1 \in \mathcal{O}_1$ (this follows from the fact that $\mathcal{O}_1$ is a finite cover of X). Analogously, for every $m > 1$, we can find a subsequence $\{x_n^m\}_{n\in\mathbb{N}}$ of $\{x_n^{m-1}\}_{n\in\mathbb{N}}$ such that all its elements are contained in the same $O_m \in \mathcal{O}_m$. Finally, with a diagonal argument, we consider the sequence $\{x_k^k\}_{k\in\mathbb{N}}$ (which is a subsequence of $\{x_n\}_{n\in\mathbb{N}}$). Observe that $\{x_k^k\}_{k\in\mathbb{N}}$ is eventually contained in $O_m \in \mathcal{O}_m$ for every $m \in \mathbb{N}$. Hence in particular $\{x_k^k\}_{k\in\mathbb{N}}$ is a Cauchy sequence, because O_m is a metric ball of radius $1/m$. Therefore $\{x_k^k\}_{k\in\mathbb{N}}$

converges to some point $x \in X$ by completeness of X. This proves that $\{x_n\}_{n\in\mathbb{N}}$ has a converging subsequence and thus that X is sequentially compact. □

Corollary 3. *A subset $C \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Proof. By the equivalence "(C1) $\iff$ (C3)" in the previous proposition we have that $C \subset \mathbb{R}^n$ is compact if and only if it is complete and totally bounded. Then the result follows from the Claims 1 and 2 below.

Claim 1. Let (X, d) be a complete metric space. Then a subset $Y \subseteq X$ is closed if and only if it is complete. Observe that this applies in particular to $X = \mathbb{R}^n$ with the Euclidean distance.

Proof. First assume that $Y \subseteq X$ is closed and consider a Cauchy sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq Y$, which converges to some point $x \in X$ by completeness of X. Hence, since Y is closed and $x_n \in Y$ for every $n \in \mathbb{N}$, x is contained in Y too, which proves that Y is complete.

Viceversa, assume that $Y \subseteq X$ is complete and consider a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq Y$ converging to some point $x \in X$. Then, for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(x_n, x) \leq \varepsilon/2$ for every $n \geq N$. As a result we obtain that $d(x_n, x_m) \leq d(x_n, x) + d(x_m, x) \leq \varepsilon$ for every $n, m \geq N$, which proves that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Therefore this sequence converges to some point $y \in Y$ by completeness of Y. However, notice that y must coincide with x since the limit of a sequence in a metric space is unique. Hence we have shown that $x \in Y$, so Y is closed. □

Claim 2. A subset of $\mathbb{R}^n$ is totally bounded if and only if it is bounded.

Proof. Consider a subset Y of $\mathbb{R}^n$. Obviously if Y is totally bounded then it is bounded. Indeed, there exist $x_1, \dots, x_k \subseteq Y$ such that $Y \subseteq \cup_{i=1}^k B(x_i, 1)$. Therefore for every $x, y \in Y$ we have that $|x - y| \leq 2 + \max_{i,j=1\dots,k} |x_i - x_j| < \infty$.

For the other implication, first observe that a subset Y of a totally bounded space (X, d) is totally bounded. Indeed, given any $\varepsilon > 0$, there exist $x_1, \dots, x_k \in X$ such that $X \subseteq \cup_{i=1}^k B(x_i, \varepsilon/2)$. Then, for every $i = 1, \dots, k$, choose $y_i \in Y \cap B(x_i, \varepsilon/2)$ (if it exists, otherwise we just ignore the index). We claim that $Y \subseteq \cup_{i=1}^k B(y_i, \varepsilon)$. This follows from the fact that $B(x_i, \varepsilon/2) \subseteq B(y_i, \varepsilon)$ (you can check it, using the triangle inequality).

Given this preliminary fact, we can now prove easily that if $Y \subseteq \mathbb{R}^n$ is bounded then it is totally bounded. Indeed, by boundedness of Y, there exists $R > 0$ such that $Y \subseteq [-R, R]^n$ and we will now show that $[-R, R]^n \subseteq \mathbb{R}^n$ is totally bounded. Taken any $\varepsilon > 0$, we cover $[-R, R]^n$ with a finite number of cubes $C_1, \dots, C_k$ with edges of length less that $2\varepsilon/\sqrt{n}$ (this is easily obtained by covering the interval $[-R, R]$ with a finite number of intervals of length less than $2\varepsilon/\sqrt{n}$ and then considering the "product cover"). Then denote by $x_1, \dots, x_k$ the center of the cubes and observe that $C_i \subseteq B(x_i, \varepsilon)$ for every $i = 1, \dots, k$, since the diameter of C_i is $\sqrt{n} \cdot 2\varepsilon/\sqrt{n}$. Therefore $[-R, R]^n \subseteq \cup_{i=1}^k C_i \subseteq \cup_{i=1}^k B(x_i, \varepsilon)$, which proves the total boundedness of $[-R, R]^n$ by arbitrariness of $\varepsilon > 0$. □

Note that these two claims conclude the proof, since Claim 1 shows that $C \subset \mathbb{R}^n$ is closed if and only if it is complete and Claim 2 proves that $C \subset \mathbb{R}^n$ is bounded if and only if it is totally bounded. Hence $C \subset \mathbb{R}^n$ is compact if and only if it is complete and totally bounded, if and only if it is closed and bounded. □

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