CANTOR SET

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Review of compactness

Def X is compact if every open cover of X admits a finite subcover.

Example If X is finite, then it is cpt

The converse is not true, but a cpt space behaves under many perspectives as a finite space ("almost finite")

For example, given X cpt, we have that:

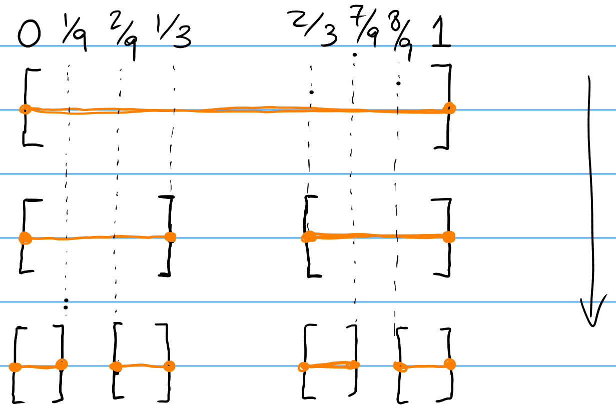
- $f: X \to \mathbb{R}$ cont, f is bdd
- $f: X \to \mathbb{R}$ cont, f admits min/max
  $(f(X) = [a,b])$
- every seq. in X admits an accumulation point.
  [In general cpt $\not\Rightarrow$ sequentially cpt]

Cantor set $\quad C \subseteq [0,1] \subseteq \mathbb{R}$

$C_0 = [0,1]$

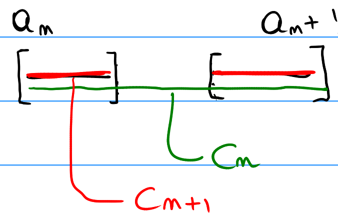
$C_1 = [0,\frac{1}{3}] \cup [\frac{2}{3},1]$

$C_2 = [0,\frac{1}{9}] \cup [\frac{2}{9},\frac{1}{3}] \cup [\frac{2}{3},\frac{7}{9}] \cup [\frac{8}{9},1]$



At each step I have C_n, which is the (disj) union of 2^n closed intervals of length $\frac{1}{3^n}$.

We construct C_{n+1} from C_n: given $I_n = [a_n, a_n + \frac{1}{3^n}] \subseteq C_n$ we put in C_{n+1} the intervals $[a_n, a_n + \frac{1}{3^{n+1}}]$ and $[a_n + \frac{2}{3^{n+1}}, a_n + \frac{1}{3^n}]$



Cantor set $\quad C := \bigcap_{n=0}^{\infty} C_n$.

Observe $\quad C_0 \supseteq C_1 \supseteq C_2 \supseteq \dots$

Properties:

- C is non-empty. $\quad 0,1 \in \underset{[0,1/3^n]}{\underset{\cup\!\mid}{C_n}} \quad \forall n$
  $\Rightarrow 0,1 \in C$
  Given an interval $I_n \subseteq C_n$ of the procedure $(I_n = [a_n, a_n + \frac{1}{3^n}]) \Rightarrow$ the end points of I_n $(a_n, a_n + 1/3^n)$ are contained in $C_k \;\forall k$, C.
- C is closed (intersection of closed sets)
  $\Rightarrow C$ is cpt (closed + bdd)

- C is totally disconnected

  $\hookrightarrow$ every $A \subseteq C$ connected is empty or $A = \{x\}$

[Prob. 3.7, $C \subseteq \mathbb{R}$ is tot. disc. iff C does not contain non-empty open interval.]

<u>Proof</u> Assume by contr. that $\exists\ (a,b) \subseteq C$ with $a<b$.

Then $(a,b) \subseteq C_n \quad \forall n \in \mathbb{N}$.

But the connected components of C_n are the intervals $\underbrace{[0, 1/3^n], [2/3^n, 1/3^{n-1}], \dots [1-\frac{1}{3^n}, 1]}_{2^n}$.

So (a,b) is contained in one of these intervals

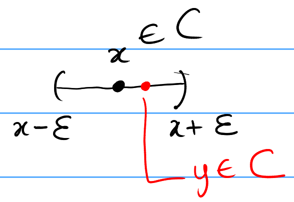
$\Rightarrow \quad b-a = |(a,b)| \leq 1/3^n$

$\hookrightarrow$ this must hold $\forall n$.

Contradiction!

$\Longrightarrow \quad C$ tot. disc.

- C has no isolated points. $\begin{pmatrix} \forall x \in C \ \forall U \text{ open neighb of } x \\ U \cap (C \setminus \{x\}) \neq \emptyset \end{pmatrix}$

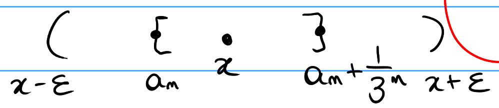


Let $x \in C$ and $\varepsilon > 0$ $(U = (x-\varepsilon, x+\varepsilon))$. We want to find $y \in (C\setminus\{x\}) \cap (x-\varepsilon, x+\varepsilon)$.

Take n s.t. $\varepsilon > \frac{2}{3^n}$, then $x \in C_n = \bigcup$ closed intervals of length $1/3^n$

$\Rightarrow x \in$ one of these intervals $[a_n, a_n + \frac{1}{3^n}]$

$a_n, a_n + \frac{1}{3^n} \in C$ and either $a_n \neq x$ or $a_n + \frac{1}{3^n} \neq x$



$[a_n, a_n+\frac{1}{3^n}] \subseteq (x-\varepsilon, x+\varepsilon)$

If we take $y = \begin{cases} a_n & \text{if } a_n \neq x \\ a_n + \frac{1}{3^n} & \text{if } a_n + \frac{1}{3^n} \neq x \end{cases}$ we are done $(y \in (C\setminus\{x\}) \cap (x-\varepsilon, x+\varepsilon))$

<u>Notation</u> C is perfect in $\mathbb{R}$ (closed + no isolated pts)

Another characterisation of the Cantor set as the pts in $[0,1]$ with digits 0, 2 in base 3.

$$\begin{array}{cccc} 0 & 1/3 & 2/3 & 1 \\ [&](&)[&] \end{array}$$

$x = 0.0\_\_\_$

$x = 0.1\_\_\_$

$x = 0.2\_\_\_$

$x \in [0,1]$

In base 3

$x = 0.2012010002\ldots$

Observe $\frac{1}{3} = 0.1 = \underbrace{0.0222\ldots\ldots}$, $\frac{2}{3} = 0.2$, $1 = 1 = 0.222\ldots$

Then $\left(\frac{1}{3}, \frac{2}{3}\right)$ consists of all the points in $[0,1]$ that have base 3 representation $0.1\underbrace{\_\_\_}_{\neq 0}$

$$\begin{array}{cccc} 0 & 1/9 & 2/9 & 1/3 \\ [&](&)[&] \end{array}$$

$\hookrightarrow$ Points with base 3 repr. $= 0.01\underbrace{\_\_\_}_{\neq 0}$

At the nth step we remove the points that have 1 as n-digit in base 3.

$$\Rightarrow C = \left\{ x \in [0,1] : \begin{array}{c}\text{the base 3 repr of } x \text{ has} \\ \text{only digits } 0, 2\end{array} \right\}$$

Note that the end points of the intervals are the points with base 3 repr definitely 0 or 2.

$\frac{1}{3} = 0.02222\ldots$ $\frac{2}{3} = 0.20000\ldots$

<u>Remark</u> $0.212222\ldots = 0.22 \in C$

If we consider the repr. with digits 0,2, the representation is unique.

Prop. $C \underset{\text{homeo}}{\cong} \{0,2\}^{\mathbb{N}}$

↳ cpt (lecture 11)

↳ the topology is the infinite product topology.

A basis is $\{\{a_1\}\times\{a_2\}\times\dots\times\{a_k\}\times\{0,2\}\times\dots\times\{0,2\}\times\dots\}$

$a_1,\dots,a_k = 0,2.$

[In general a basis for $\prod_{i\in\mathbb{N}} X_i$ is $\underset{\substack{\downarrow \\ \text{open subset of } X_1}}{U_1}\times\dots\times \underset{\substack{\llcorner \\ \text{open subset of } X_k}}{U_k}\times X_{k+1}\times X_{k+2}\times\dots$]

$$\mathcal{B} := \{\{a_1\}\times\dots\times\{a_k\}\times\{0,2\}\times\{0,2\}\times\dots \mid a_1,\dots,a_k = 0,2\} \quad \text{(EX)}$$

is a basis for the topology of $\{0,2\}^{\mathbb{N}}$.

$$f : C \longrightarrow \{0,2\}^{\mathbb{N}}$$
$$x\in[0,1] \longmapsto (a_1,a_2,a_3,\dots) \qquad \text{where } x = 0.a_1a_2a_3\dots$$

We said before that f is a bijection.

We have to prove that f is a homeom. (cont + open).

<u>Continuous</u>: consider $U := \{a_1\}\times\dots\times\{a_k\}\times\{0,2\}\times\{0,2\}\times\dots \in \mathcal{B}$

$$f^{-1}(U) = \{x\in C : x = 0.a_1\dots a_k\,\underline{\quad\quad}\} \text{ open in } C.$$

$$I_k := \big(\,[\,0.a_1\dots a_k,\ 0.a_1\dots a_k22222\dots\,]\,\big)$$

↳ end points of an intervals in C_k ↲

$I_k\cap C$ is open in C and $I_k\cap C = f^{-1}(U)$.

<u>Open</u>: <u>EX</u>