Functional Analysis 1 — L1 – 17/9/2020

what? study of ∞-dim'l vector spaces (over $\mathbb{R}$ or $\mathbb{C}$) and linear maps between them.

Def. V is an ∞-dim vector space $/K$ such that $\forall N \in \mathbb{N} = \{0,1,2,3,\dots\}$ there are N lin. indep. vectors in V.

Examples:

$C^0([0,1])$
$L^p(-\pi, 2\pi)$
$W^{k,q}(-20, 17)$

rough idea "spaces whose points are function"

let's check this is indeed an ∞-dim vector space over $\mathbb{R}$.

$V := C^0([0,1])$

$V_n := \langle 1, x, x^2, \dots, x^n \rangle_{\mathbb{R}}$

claim $\{1, x, x^2, \dots, x^n\}$ lin. indep.

$$\sum_{i=0}^{n} \lambda_i x^i = 0 \quad (*)$$

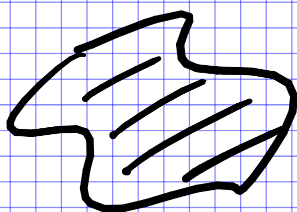
- evaluate $(*)$ at $\bar{x}=0 \Rightarrow \lambda_0 = 0$
- then take $\frac{d}{dx}$ of $(*)$, evaluate at $\bar{x}=0 \Rightarrow \lambda_1 = 0$
- und so weiter $\Rightarrow \dots \lambda_n = 0$ $\square$

<u>why ?</u> both in pure and applied Math, there are lots of situations where ∞-dim vector spaces naturally arise.

- study of partial diff. equations, e.g.

$$\begin{cases} \boxed{\Delta u = f} & \Omega \\ u = 0 & \end{cases} \qquad (\$)$$

gravitational potential u associated to mass density f



Spaces where to "look for solutions" are indeed of the type above.
Same story for e.g.

$$u_t = \Delta_x u \qquad \text{heat eq.}$$
$$u_{tt} = \Delta_x u \qquad \text{wave eq.}$$

etc.... classical physics

- mathematical foundations of <u>quantum mechanics</u>: basic concepts like e.g. state, observable etc... are given in terms of vectors in Hilbert spaces, and linear (bounded / unbounded) operators.
From there: language of quantum field theory.
<u>XX^{th} century Physics</u>

Some things "work as in the finite-dim. case" but others are much different, abstract and counterintuitive. (examples (1) and (2) below).

Also: generalisation principle

→ we don't study spaces "one at a time", but build a theory for all spaces that have the same structures.

how? the central object of the course are $\boxed{\text{Banach spaces}}$: def complete normed space

Hilbert spaces		Banach spaces		complete metric spaces
$(V, \langle\cdot,\cdot\rangle)$ req. complete	$\subset$	$(V, \|\cdot\|)$ req.: complete	$\subset$	(V, d) req. complete

recall:

- a scalar product induces a norm
$$\|x\| := \sqrt{\langle x, x\rangle}$$
- a norm induces a distance
$$d(x_1, x_2) := \|x_1 - x_2\|$$
- a distance induces a topology
$$\Omega \subset V \text{ open} \iff \forall x \in \Omega \ \exists r > 0 \text{ w/ } B_r(x) \subset \Omega$$

Achtung! the inclusions above are "very much strict" in the ∞-dim context.

E.g. (exercise 1.3): we'll see that

$\underbrace{L^p(0,1)}_{}$ are Hilbert IFF $\boxed{p=2}$.

Banach spaces $\quad \|u\|_{L^p} = \left(\int |u|^p dx\right)^{1/p}$

Two "prototypical" examples:

1. **Riesz representation theorem**

$(V, \langle \cdot, \cdot \rangle) \rightsquigarrow$ isomorphism

$$V \cong V^*$$

⌜ $\forall \ell \in V^* \ \exists! v \in V$ such that

$$\boxed{\ell(w) = \langle v, w \rangle}$$

$\forall w \in V$ ⌟

Same theorem (w/ harder proof) for Hilbert spaces.

2. the Heine-Borel theorem

($\overline{B_1}(0)$ compact in Euclidean $\mathbb{R}^n$)

This is much false in ∞-dim: In a Banach space X the unit ball

$$B = \{ x \in X : \ \|x\| < 1 \}$$

is relatively compact (i.e. it has compact closure) if and only if X has finite dimension.

(e.g. $L^p(0,1)$ has non-compact unit ball $\forall p \in [1, \infty]$.)

$\hookrightarrow$ motivation to: weak topologies (...)

Resources:

- website
- lectures and videos
- notes / textbook(s)
- office hours (Thursdays 4-6 pm)
- D-Math forum
- problem sets
- exercise classes

The team:

Giada Franz (course coordinator)

Riccardo Caniato
Federico Franceschini
Filippo Gaia

exercise classes in presence

Salome Schumacher

exercise class online

Study advice :

i) Math is <u>not</u> a spectator sport
(pro-active attitude in class)

ii) " you don't learn how to play football by reading a book about football "
(homework is key)

iii) build an athlete-type weekly routine

Expect the best and give me your best !