Functional Analysis I - L2 21/9/2020

Metric spaces and Baire thm.

Def. (X,d) is called a metric space if X is a set, $d: X\times X \longrightarrow \mathbb{R}$ satisfying:

i) $d(x_1,x_2)\geq 0 \;\; \forall\, x_1,x_2$ and $d(x_1,x_2)=0 \iff x_1=x_2$

ii) $d(x_1,x_2)=d(x_2,x_1) \;\; \forall x_1,x_2$

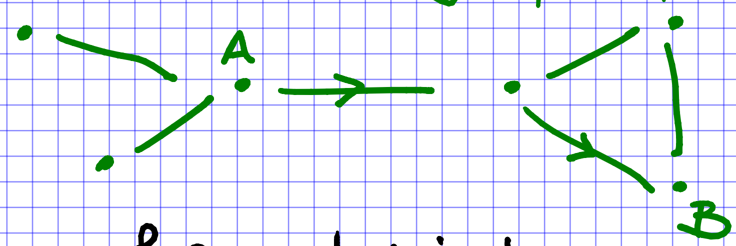
iii) $d(x_1,x_3)\leq d(x_1,x_2)+d(x_2,x_3) \;\; \forall x_1,x_2,x_3$.

Example:

1. $(V,\|\cdot\|)$ linear normed space. It inherits a metric structure by simply declaring $d(v_1,v_2)=\|v_1-v_2\|$.

2. $S^n \subset \mathbb{R}^{n+1}$
   $= \{x\in\mathbb{R}^{n+1} : \|x\|=1\}$
   "A subset of a metric space is a metric space."

3. X a connected graph (possibly infinite)



e.g. $d(A,B)=2$.

Review of some terminology:

- $Y\subset X$ open if $y\in Y \Rightarrow \exists\, r>0$ s.t. $B_r(y)\subset Y$
- $Y\subset X$ closed if ${}^cY := X\setminus Y$ is open
- interior of a set $\mathring{Y} = \bigcup_{\substack{Z\subset Y \\ Z \text{ open}}} Z$

- closure of a set $\overline{Y} = \bigcap\limits_{\substack{Z \supset Y \\ Z \text{ closed}}} Z$
- boundary of a set $\partial Y = \overline{Y} \setminus \mathring{Y}$

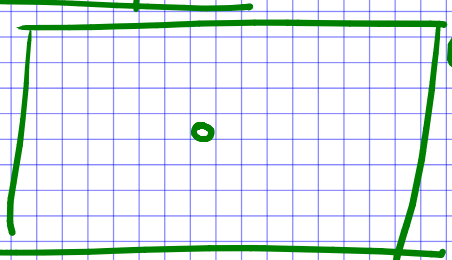
Rmk. equivalent char. given in the Topology course.

Completeness Given a metric space (X, d) we say it is complete if any Cauchy sequence is converging.

recall:

- a sequence (x_n) is Cauchy if $\forall \varepsilon > 0 \ \exists \, n_0 = n_0(\varepsilon)$ s.t. $\forall n, m \geq n_0 \quad d(x_n, x_m) < \varepsilon$.
- a sequence (x_n) is convergent if $\exists \, x \in X$ s.t. $d(x_n, x) \xrightarrow[n \to \infty]{} 0$

Example: a non-complete metric space.



$(\mathbb{R}^2, \text{Euclidean})$

remove the origin $\rightsquigarrow$ "sequences converging to the origin in $\mathbb{R}^2$ are not converging anymore"

Exercise: any graph w/ path distance is complete. (Hint: what can you say about a Cauchy sequence there...).

Examples:

$$\left.\begin{array}{l} C^k([0,1]) \\ L^p(\Omega) \end{array}\right\} \underbrace{\text{complete normed spaces}}_{\text{Banach spaces}}$$

Towards the statement of the Baire theorem

(X, d) metric spaces

- $Y \subset X$ **dense** if any of the following 3 equivalent conditions is true:

  1) $\forall\ B = B_r(x) \subset X \quad B \cap Y \neq \emptyset$
  2) $\forall\ \mathcal{U} \neq \emptyset \subset X$ open $\quad \mathcal{U} \cap Y \neq \emptyset$
  3) $\overline{Y} = X$.

Examples: • $\mathbb{Q} \subset \mathbb{R}$

• $\mathbb{Q}^F_\delta = \bigcup_{k \in \mathbb{N}} (q_k - \delta 2^{-k}, q_k + \delta 2^{-k})$

- $Y \subset X$ **nowhere dense** if any of the following 3 equivalent conditions is true:

  1) $\forall\ B = B_r(x) \subset X \quad B \setminus \overline{Y} \neq \emptyset$
  2) $\forall\ \mathcal{U} \neq \emptyset \subset X$ open $\quad \mathcal{U} \setminus \overline{Y} \neq \emptyset$
  3) $\overset{\circ}{\overline{Y}} = \emptyset$

(morally speaking: $\overline{Y}$ contains no open ball)

rmk. if $Y \subset X$ closed, have that:

Y is nowhere dense $\iff$ Y has empty interior.

Example: $\underbrace{\mathbb{R} \setminus \mathbb{Q}^F_\delta}$ closed, empty interior $\Rightarrow$ by previous remark $\rightarrow$ is nowhere dense.

Easy proposition (complementing game):

Let (X, d) be a metric space. Then TFAE:

i) $Y \subset X$ open and dense

ii) $Y^c \subset X$ closed and nowhere dense.

Proof. i) $\Rightarrow$ ii) pick any ball $B \subset X$

$$B \setminus \overline{Y^c} = B \setminus Y^c = B \cap Y \neq \emptyset .$$

ii) $\Rightarrow$ i) $Y = (Y^c)^c \rightsquigarrow Y$ open. pick any ball $B \subset X$

$$B \cap Y = B \setminus Y^c = B \setminus \overline{Y^c} \neq \emptyset . \quad \blacksquare$$

Baire theorem: For a complete metric space (X, d) the following (equivalent) assertions are true:

i) $(U_j)_{j \in \mathbb{N}}$ open, dense sets in X, then

$$U := \bigcap_{j \in \mathbb{N}} U_j \quad \text{dense}$$

ii) $(A_j)_{j \in \mathbb{N}}$ closed, w/ empty interior in X, then

$$A := \bigcup_{j \in \mathbb{N}} A_j \quad \text{has empty interior.}$$

Comments on the assumptions:

- these are countable operations

(else: $(X, d) = (\mathbb{R}, |\cdot|)$ $\quad \underbrace{A_x = \{x\}}_{\text{closed, nowhere dense}}$ $\quad x \in \mathbb{R}$

but $\bigcup_{x \in \mathbb{R}} A_x = \mathbb{R}$)

- the topological assumption (open for part i)) is crucial

else pick $U_k := \mathbb{Q} + k\pi \qquad k \in \mathbb{Z}$

dense

but we can check that $\bigcap_k U_k = \emptyset$. In fact, any pairwise intersection is empty:

$$q_1 + k_1\pi = q_2 + k_2\pi$$

$$(k_2 - k_1)\pi = q_1 - q_2$$

$$\rightsquigarrow \quad \pi = \frac{q_1 - q_2}{k_2 - k_1} \in \mathbb{Q}.$$

Proof.

claim: U is dense i.e. $\forall B = B_r(x) \quad B \cap U \neq \emptyset$

toy model: "finite intersection of open, dense sets is dense"

U_1, U_2 open, dense $\qquad \forall B \quad \underbrace{B \cap (U_1 \cap U_2)}_{} \overset{?}{\neq} \emptyset$

$\downarrow$ (?)

$U_1 \cap U_2$ dense $\qquad \underbrace{(B \cap U_1)}_{\text{open}} \cap U_2 \neq \emptyset$

invoke its density

general case: inductive construction of a sequence $(x_j)_{j \in \mathbb{N}}$.

a) given B (by the enemy)

$$U_1 \text{ open and dense} \implies \underbrace{B \cap U_1}_{\ni x_1} \neq \emptyset$$

$$\exists\, r_1 \underline{\in (0, 1/2)} \text{ s.t.}$$

$$B_{r_1}(x_1) \subset B_{2r_1}(x_1) \subset B \cap U_1$$

Now set $B_j := B_{r_j}(x_j)$, where $B_0 = B$

and proceed inductively.

b) chosen $x_1, \dots, x_{j-1} \in X$

$r_1, \dots, r_{j-1} \in \mathbb{R}_{>0}$

U_j open and dense $\rightsquigarrow$ $\underbrace{B_{j-1} \cap U_j}_{\ni x_j} \neq \emptyset$

$\exists\ r_j \in (0, 2^{-j})$ s.t. $\overline{B_{r_j}(x_j)} \subset B_{2r_j}(x_j) \subset U_j \cap B_{j-1}$

We get a nested family of balls, in part. $\forall j > k$

$B_{r_j}(x_j) \subset U_j \cap B_{r_{j-1}}(x_{j-1}) \subset B_{r_{j-1}}(x_{j-1})$

$\subset \dots \subset B_{r_k}(x_k)$

$d(x_j, x_k) < 2^{-k} \Longrightarrow (x_j)$ Cauchy

by completeness of $(X, d) \Longrightarrow x^* = \lim_{j\to\infty} x_j$

Claim: $\boxed{x^* \in U \cap B}$

By construction the set $S_k := \{x_j : j > k\}$

k-tail of the sequence

$S_k \subset B_k \equiv B_{r_k}(x_k)$

$\overline{S_k} \subset \overline{B_k}$ but by convergence of the sequence

$\overline{S_k} = S_k \cup \{x^*\}$

$x^* \in \overline{B_k} \subset B_{2r_k}(x_k) \subset U_k \quad (\forall k)$

$\Longrightarrow x^* \in U = \bigcap_k U_k$

Also: $S_1 \subset B_1 = B_{r_1}(x_1)$ hence

$$\overline{S_1} \subset \overline{B_1} \subset B_{2r_1}(x_1) \subset U_1 \cap B \subset B$$

$$\Longrightarrow \quad x^* \in B$$

Conclusion: $U \cap B \neq \emptyset$. ∎

How to use Baire? "contrapositive of ii)"

$$[a \Rightarrow b] \iff [\neg b \Rightarrow \neg a]$$

example: if Plato is a human being, then he is mortal.

TRUE if and only if

"if Plato is not mortal, then he is not a human being"

So ii) $\iff$ ii)' which goes as follows

(X, d) complete metric space. (F_n) closed sets

If $F := \bigcup F_n$ has non-empty interior then

$\exists \, n_*$ s.t. $\mathring{F}_{n_*} \neq \emptyset$.

In fact, the typical use is:

$$(*) \quad \boxed{F = X \Longrightarrow \exists \, n_* \text{ s.t. } \mathring{F}_{n_*} \neq \emptyset}$$

Example let (X, d) complete metric space

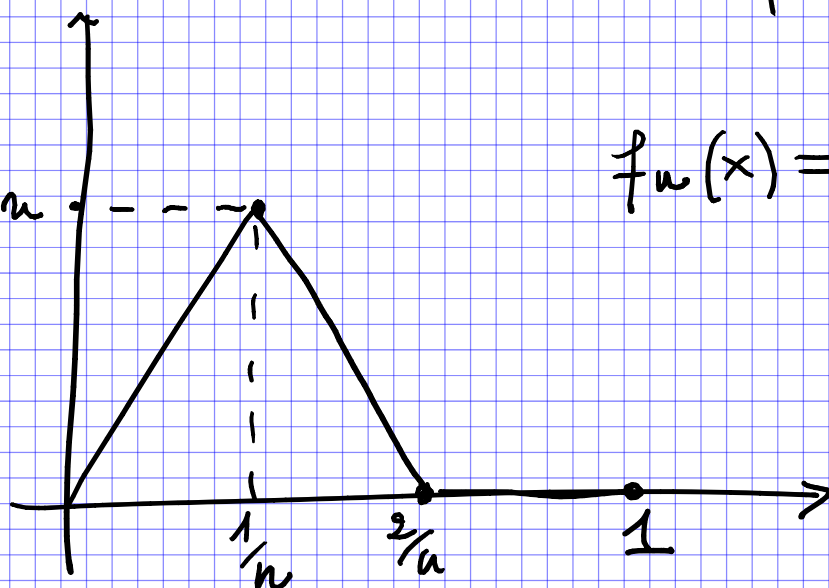
$(f_\lambda)_{\lambda \in \Lambda} \subset C^0(X)$. Suppose $(f_\lambda)_{\lambda \in \Lambda}$ is pointwise bounded, i.e. $\forall x \in X$

$$\boxed{\sup_{\lambda \in \Lambda} |f_\lambda(x)| < \infty}$$

_9. Is (f_λ) uniformly bounded ?

(i.e. $\sup_{\lambda \in \Lambda} \sup_{x \in X} |f_\lambda(x)| < \infty$)

Answer: NO. Counterexample:



$$f_n(x) = \begin{cases} n^2 x & x \in [0, 1/n] \\ 2n - n^2 x & x \in [1/n, 2/n] \\ 0 & x \in [2/n, 1] \end{cases}$$

Claim 1 : (f_n) is pointwise bounded.

Claim 2 : $\max_{x \in [0,1]} |f_n(x)| = n$

$\Longrightarrow \sup_{n \in \mathbb{N}} \left(\uparrow \right) = +\infty \Longrightarrow$ no uniform bound

make-up question : can we find an open set B (open interval in the example)

such that $\sup_{\lambda \in \Lambda} \sup_{x \in \overline{B}} |f_\lambda(x)| < \infty$?

Answer: YES, using Baire in form $(*)$

Given $k \in \mathbb{N}$, set $A_k := \{ x \in X : \sup_{\lambda \in \Lambda} |f_\lambda(x)| \leq k \}$

closed in X

$\bigcup_k A_k = X$ by hp. pointwise bound

if $\sup_{\lambda\in\Lambda} |f_\lambda(x)| =: c(x)$

then $x \in A_{\lceil c(x)\rceil}$

$\underset{\text{Baire}}{\leadsto}$ $\exists\ k_* \in \mathbb{N}$ w/ $\mathring{A}_{k_*} \neq \emptyset$ which means

$\exists\ B \subset X$ ball, $B \subset \mathring{A}_{k_*}$ where

$(f_\lambda)_{\lambda\in\Lambda}$ is uniformly bounded. ∎

Next time: we'll use ↑ to prove Banach-Steinhaus

i.e. uniform boundedness for lin. operators between Banach spaces. ∎