Functional Analysis I - L3 24/9/2020

Uniform boundedness for linear maps (i.e. Banach-Steinhaus thm)

Let $(V, \|\cdot\|)$ complete normed space (i.e. Banach space), and let (T_λ) be a family of linear maps $T_\lambda : V \to \mathbb{R}$ assume $\forall v \in V$, $\sup_{\lambda \in \Lambda} |T_\lambda(v)| < \infty$. Then the family (T_λ) is uniformly continuous in the sense that $\exists C > 0$ s.t. $\boxed{|T_\lambda(v)| \leq C\|v\|}$ $\forall v \in V$.

uniform bound on the operator norm of (T_λ)

Proof. by last time (application (1) of Baire) $\exists$ ball $B_r(w) \subset V$ s.t. $\sup_{\lambda \in \Lambda} \sup_{x \in B_r(w)} |T_\lambda(x)| \leq \bar{C}$

$$\iff \sup_{\lambda \in \Lambda} \sup_{v \in B_r(0)} |T_\lambda(w+v)| \leq \bar{C}$$

$$|T_\lambda(v)| = |T_\lambda(w+v) - T_\lambda(w)|$$

$$\overset{\text{triangle ineq.}}{\leq} |T_\lambda(w+v)| + |T_\lambda(w)|$$

$$\Rightarrow \sup_{\lambda \in \Lambda} \sup_{v \in B_r(0)} |T_\lambda(v)| \leq C \quad (< \infty) \qquad (*)$$

Now, we play w/ scaling: given any $v \neq 0$, $v \in V$

$$\tilde{v} = \frac{r}{2} \frac{v}{\|v\|} \quad \text{and so} \quad \tilde{v} \in B_r(0)$$

Apply $(*)$ to $\tilde{v}$: $\sup_{\lambda \in \Lambda} |T_\lambda(\tilde{v})| \leq C$

$$\parallel$$

$$\sup_{\lambda \in \Lambda} \frac{r}{2\|v\|} |T_\lambda(v)| \leq C$$

$$\iff \sup_{\lambda\in\Lambda} |T_\lambda(v)| \le \frac{2C}{r} \|v\|$$

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Algebraic basis

<u>Def.</u> Let X be a vector space over a field $\mathbb{K}$. We say that a subset $E\subset X$ is an <u>algebraic basis</u> for X if any vector $x\in X$ can be written uniquely as a <u>finite</u> linear combination of elements in E w/ coefficients in $\mathbb{K}$.

<u>Thm.</u> Let $(X, \|\cdot\|)$ be a complete normed space. Then any algebraic basis of X is either <u>finite</u> or uncountable.

(moral: there are, as we'll see, better notions of basis . . . .)

<u>Rmk. 1</u>: when we say <u>normed</u> vector space we tacitly mean that $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$. $(\|\cdot\|: V\longrightarrow \mathbb{R})$.

<u>Rmk. 2</u>: we give for granted that "any vector space admits an algebraic basis". Idea: same story as 1st year but one needs Zorn's Lemma.

<u>Proof.</u> take E a countable basis of X: we want to prove it is actually finite. Let $E=\{e_1, e_2, \dots\}$, set

$$A_n := \langle e_1, e_2, \dots, e_n\rangle \qquad n\in\mathbb{N}_*$$

Since E is an algebraic basis $\Longrightarrow \bigcup_n A_n = X$

<u>Claim 1</u>: $A_n\subset X$ is <u>closed</u>.

- a subspace of a complete metric space is complete $\iff$ closed (i.e. it is enough for us to check that $(A_n, \|\cdot\|)$ complete)

- now consider on A_n two norms: $\|\cdot\|$ restriction of norm on X

$$\left\| \sum_{i=1}^{n} \lambda_i e_i \right\|' = \sqrt{\sum_{i=1}^{n} |\lambda_i|^2}$$

by (cf. Exercise 1.1) these two are equivalent i.e. $\exists C>0$

$$\boxed{C^{-1}\|v\| \leq \|v\|' \leq C\|v\|} \quad \forall v \in A_n.$$

$\Longrightarrow$ for any sequence (x_k) in A_n

(x_k) is Cauchy in $(A_n, \|\cdot\|) \Longleftrightarrow (x_k)$ is Cauchy in $(A_n, \|\cdot\|')$

(x_n) converges in $(A_n, \|\cdot\|) \Longleftrightarrow (x_n)$ converges in $(A_n, \|\cdot\|')$

Conclusion: since $(A_n, \|\cdot\|')$ complete (since $\mathbb{R}^n$ is complete!) then $(A_n, \|\cdot\|)$ complete. ∎

<u>Claim 2</u>: $\mathring{A}_n \neq \emptyset \Longrightarrow E$ is finite and $\dim X < \infty$

$\exists\, \epsilon > 0 \;\; B_\epsilon(x) \subset A_n$ $(\exists x \in A_n)$ $\Big\} \rightarrow B_\epsilon(0) = B_\epsilon(x) - x \subset A_n$

$$A_n \supset \{\lambda y : \lambda > 0,\ y \in B_\epsilon(0)\} = X$$

$\Longrightarrow \dim X = \dim A_n = n$ ∎

Claim 1 + Claim 2 + Baire $\Longrightarrow$ conclusion.

(more precisely: Baire leads to a contradiction unless $\dim X < \infty$)

∎

Achtung! COMPLETENESS is crucial.

$X = \{$ polynomials $p : [0,1] \longrightarrow \mathbb{R} \}$

an algebraic basis for X is (trivially)

$$\{ f_n = x^n \, , \, n \in \mathbb{N} \} = E$$

Now, take on X the C^0-norm, i.e.

$$\| f \|_{C^0} = \sup_{x \in [0,1]} |f(x)|$$

By previous thm. we MUST conclude that

$$(X, \|\cdot\|_{C^0}) \text{ is } \underline{\text{not}} \text{ complete.}$$

-? What is the metric completion of $(X, \|\cdot\|_{C^0})$?

(hint : what is the closure of $X \subset C^0([0,1])$).

Baire Category

Def. (X, d) metric space.

i) $A \subset X$ is called meager (or 1^{st} category) if $A = \bigcup_{j \geq 1} A_j$

where each A_j is nowhere dense ; $Kat(A) = 1$.

$\Omega \subset X$ residual if $\Omega^c = X \setminus \Omega$ is meager.

ii) all sets that are not 1^{st} category are called 2^{nd} category.

Example $\mathbb{Q} \subset \mathbb{R}$ meager (1^{st} category).

Rmk.
$$\left.\begin{array}{l} Kat(A) = 1 \\ Kat(A') = 1 \end{array}\right. \Longrightarrow Kat(A \cup A') = 1.$$

Prop. Let now (X,d) be a complete metric space. Then:

1) $\operatorname{Kat}(X) = 2$

2) $\operatorname{Kat}(A) = 1 \implies \operatorname{Kat}(A^c) = 2$

A^c dense in X

3) $U \neq \phi$ open $\implies \operatorname{Kat}(U) = 2$.

Proof. 1) by contradiction $\operatorname{Kat}(X) = 1$, then by def.

$\boxed{X = \bigcup A_j}$ nowhere dense sets.

$\left(\forall j \quad \mathring{\overline{A_j}} = \phi \right)$

$\Downarrow$

$X = \bigcup \overline{A_j} \quad \overset{\text{Baire}}{\rightsquigarrow} \quad \exists\, j_0 \text{ s.t. } \mathring{\overline{A_{j_0}}} \neq \phi$

This contradiction concludes the proof.

2) use remark $\begin{cases} \operatorname{Kat}(A) = 1 \\ \operatorname{Kat}(A^c) = 1 \end{cases} \implies \operatorname{Kat}(A \cup A^c) = 1$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \operatorname{Kat}(X) = 2$ use 1)

Let's check that A^c is dense: $\boxed{A = \bigcup A_j \subset \bigcup_j \overline{A_j}}$ $\qquad A^c \supset \bigcap_j (\overline{A_j})^c$

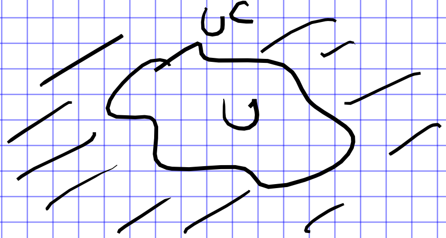
set $U_j = (\overline{A_j})^c = X \setminus \overline{A_j}$ $\qquad \equiv \underbrace{\bigcap_j U_j}_{\text{dense}}$

open, dense

(complementing game)

3) if (by contradiction) U open, $\operatorname{Kat}(U) = 1$

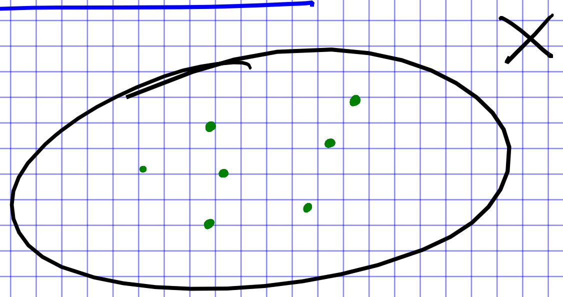
then (part 2)) $\operatorname{Kat}(U^c) = 2$, and U^c dense in X



$\implies$ contradiction

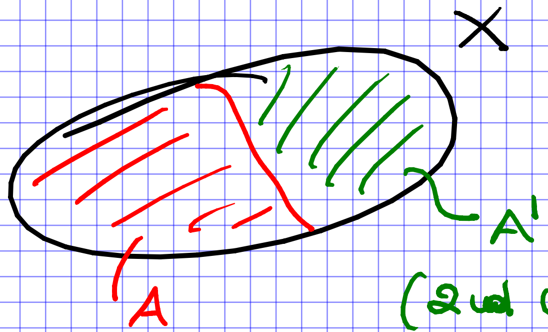
$\blacksquare$

Naive Picture



A "green dots" $\hookrightarrow$ 1st category set

$X \setminus A$ $\hookrightarrow$ 2nd category set and residual



A' (2nd cat) not residual

A (2nd cat) not residual

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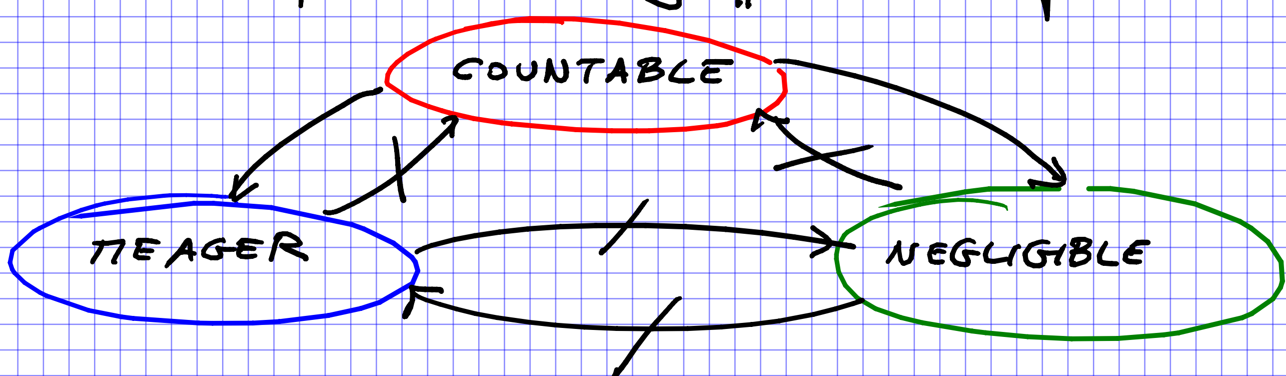
Different notions of smallness

$$A \subset (\mathbb{R}, d_{Eucl})$$

field	logic	topology	measure theory
notion of size	cardinality	Baire category	Lebesgue m.
adjective for "small"	countable	1st cat (meager)	negligible

?. mutual implications involving different notions of smallness.



- countable $\Longrightarrow$ meager TRUE

$$A = \bigcup_{n \in \mathbb{N}} \{x_n\}$$

$A_n = \{x_n\}$ closed nowhere dense $\Longrightarrow$ A meager

- countable $\Longrightarrow$ negligible TRUE

$$\mathcal{L}^1(A) = \sum_{n \in \mathbb{N}} \underbrace{\mathcal{L}^1(A_n)}_{\mathcal{L}^1(\{x_n\})} = 0$$

- negligible $\Longrightarrow$ meager FALSE

(use fattened rationals)

(q_n) numbering of $\mathbb{Q} \subset \mathbb{R}$

$$G_j := \bigcup_n \left(q_n - 2^{-(j+n+1)}, q_n + 2^{-(j+n+1)}\right)$$

open, dense

$$\mathcal{L}^1(G_j) \le \sum_n \underbrace{\mathcal{L}^1(\;)}_{2^{-(j+n)}} = 2^{1-j}$$

Set $G = \bigcap_j G_j$

dense in $\mathbb{R}$ (it contains $\mathbb{Q}$)

$$\boxed{\mathcal{L}^1(G) = 0}$$

Claim: cat $(G) = 2$

$A_j = X \setminus G_j$

closed, nowhere dense

$$\boxed{A = \bigcup A_j}$$

meager i.e. Kat $(A) = 1$

$\Longrightarrow$ Kat $(A^c) = 2$

$\qquad\qquad \shortparallel$

$\qquad$ Kat (G)

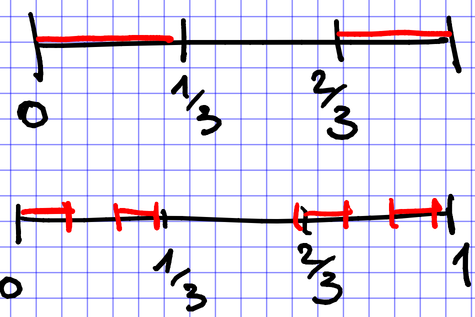
- meager $\Longrightarrow$ negligible FALSE

pick A as in previous example ✓

$\mathcal{L}^1(A) = \infty$ but A is meager

- meager $\Longrightarrow$ countable FALSE
- negligible $\Longrightarrow$ countable FALSE

$\}$ Cantor set $\subset [0,1]$



C_1

C_2

closed (compact)

$$\boxed{C = \bigcap_n C_n}$$

. . . . . . C_n 2^n closed intervals of size 3^{-n}

$$\mathcal{L}^1(C) \le \mathcal{L}^1(C_n) = \left(\tfrac{2}{3}\right)^n \xrightarrow[n\to\infty]{} 0$$

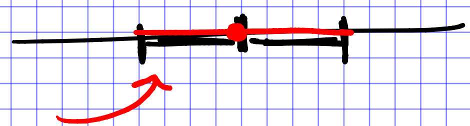
$C \underset{bij}{\sim} \{ x \in [0,1] :$ x has a ternary representation only containing digits $\{0,2\}\}$

C has same cardinality as $\mathbb{R}$ (uncountable!)

C is nowhere dense

(C has empty interior)

$(x-r, x+r) \subset C$



$k >> 1$ $3^{-k} < r$

then (if that were the picture) C would contain all real numbers whose first k digits are $\{0,2\}$ and the other ones are arbitrary! $\Longrightarrow$ C meager. ∎