Functional Analysis I - L4 28/9/2020

Structure of (dis-)continuity set

(X,d) metric space, $f: X \longrightarrow \mathbb{R}$ (no hp.)

what can we say, in general, about the set

$$C := \{x \in X : f \text{ is continuous at } x\} \quad ?$$

Motivation (Analysis 1):

(1) is there $f: \mathbb{R} \longrightarrow \mathbb{R}$ such that $C = \emptyset$?

(2) is there $f: \mathbb{R} \to \mathbb{R}$ such that $C = \mathbb{R} \setminus \mathbb{Q}$?

(3) is there $f: \mathbb{R} \to \mathbb{R}$ such that $C = \mathbb{Q}$?

Answers:

(1) is YES: Dirichlet function $f = \chi_{\mathbb{Q}} = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$

(2) is YES: ______ function $f(x) = \begin{cases} 1/q & \text{if } x = p/q \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$

(3) is NO, and this needs a general structure theorem for continuity set ⊕ Baire.

→ there is no function that is discontinuous "only" at irrationals!

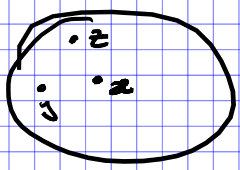
Terminology: in a top. space X (in part. in a metric space) we say that $A \subset X$ is:

- a G_δ if it is a countable intersection of open sets
- a F_σ it is is a countable union of closed sets.

Thm: Setting as above, $f: X \longrightarrow \mathbb{R}$. Then the continuity set C of f is a G_δ, i.e. there exists a sequence of open sets (O_n) in X such that $C = \bigcap_n O_n$.

Proof. Given $n \in \mathbb{N}_* = \{1, 2, \dots\}$ set

$$O_n := \{x \in X : \exists \delta > 0 \text{ w/ } \forall y, z \in B_\delta(x) : \underbrace{|f(y) - f(z)| < 1/n}_{\text{"small oscillation"}}\}$$



Claim 1: O_n is open.

$x \in O_n$, let δ be as above. We claim that $\boxed{B_{\delta/2}(x) \subset O_n}$ ($\Rightarrow$ Claim 1).

if $x' \in B_{\delta/2}(x)$ have that $\underbrace{B_{\delta/2}(x') \subset B_\delta(x)}$

$\Longrightarrow$ $\forall y, z \in B_{\delta/2}(x')$ $|f(y) - f(z)| < 1/n$ ∎

Claim 2: $C = \bigcap_n O_n$

$\boxed{C \subset \bigcap_n O_n}$ Let $x \in C$, given $n \in \mathbb{N}_*$ let $\delta > 0$ be s.t.

$$|f(x) - f(y)| < \frac{1}{2n} \quad \forall y \in B_\delta(x)$$

Taken $y, z \in B_\delta(x)$ have

$$|f(y) - f(z)| \leq |f(y) - f(x)| + |f(x) - f(z)| < \frac{1}{2n} + \frac{1}{2n} = 1/n$$

$\boxed{C \supset \bigcap_n O_n}$ Let $x \in \bigcap_n O_n$. Given $\epsilon > 0$ pick $\bar{n} \in \mathbb{N}_*$ w/ $1/\bar{n} < \epsilon$, take $\delta > 0$ s.t.

$$\boxed{\forall y, z \in B_\delta(x) \quad |f(y) - f(z)| < 1/\bar{n}}$$

take $z = x$ $\quad |f(y) - f(x)| < 1/\bar{n} < \epsilon$

$\Longrightarrow x \in C$ ∎

<u>Application</u>: there is no function $f: \mathbb{R} \to \mathbb{R}$ that is discontinuous "only" at irrationals.

<u>Proof</u>: By contradiction, if that were the case

$$\underbrace{D}_{(\mathbb{R}\setminus\mathbb{Q})} = \bigcup_n K_n = \mathbb{R}\setminus\mathbb{Q}$$

discontinuity set — structure thm. — closed sets

$$\mathbb{R} = \mathbb{Q} \sqcup (\mathbb{R}\setminus\mathbb{Q}) = \Big(\bigcup_n \{q_n\}\Big) \sqcup \Big(\bigcup_m K_m\Big)$$

countable union of closed sets and each such set has empty interior $\overset{\text{Baire}}{\Longrightarrow}$ $\mathbb{R}$ has empty interior

<u>contradiction!</u> ■

<u>A related (but harder) question:</u>

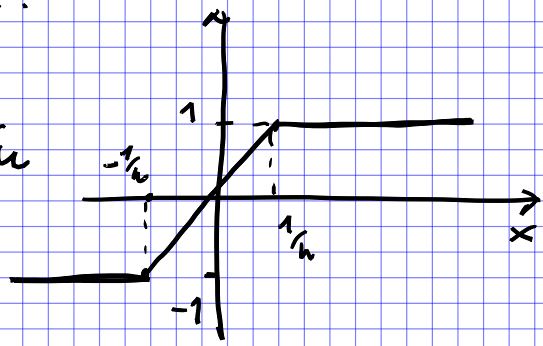
(X,d) metric space, (f_n) w/ $f_n: X \longrightarrow \mathbb{R}$ and assume that the sequence is <u>pointwise convergent</u>:

$$\forall x \in X \qquad f(x) := \lim_{n\to\infty} f_n(x)$$

_Q: (if X is complete) and f_n is C^0 $\forall n \in \mathbb{N}$
what can we say about the continuity set of the limit function f?

<u>rmk.</u> f will be discontinuous in general:

$$f_n(x) = \begin{cases} -1 & x \le -1/n \\ nx & -1/n \le x \le 1/n \\ +1 & x \ge 1/n \end{cases}$$



$$f(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ +1 & x > 0 \end{cases}$$

Thm. (exercise 2.5) the continuity set of a function arising as a pointwise limit of C^0 functions (which is always a G_δ) is a dense (G_δ) and residual in X. (i.e. it is "big")

Cor. the Dirichlet function is NOT the pointwise limit of any sequence of continuous functions.

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Banach spaces

Def. Let V be a vector space over a field $\kappa = \mathbb{R}, \mathbb{C}$.

$\|\cdot\| : V \longrightarrow \mathbb{R}$ is a norm if

1. $\|v\| \geq 0 \quad \forall v \in V, \quad \|v\| = 0 \iff v = 0$
2. $\|\lambda v\| = |\lambda| \cdot \|v\| \quad \forall \lambda \in \kappa, \ \forall v \in V$
3. $\|v + w\| \leq \|v\| + \|w\| \quad \forall v, w \in V.$

$(V, \|\cdot\|)$ is called a normed space

if complete, " " a Banach space.

rmk. : 3 $\Longrightarrow \big| \|v\| - \|w\| \big| \leq \|v - w\|.$

rmk. Banach spaces are examples of complete metric spaces (the whole story seen so far applies $\longrightarrow$ Banach-Steinhaus
$\longrightarrow$ no countable alg. bases
$\longrightarrow$ structure of cont. sets)

Examples:

a) $(\mathbb{R}^n, \|\cdot\|_{euc})$ $n \geq 1$ are Banach.

b) $(X, \mathcal{A}, \mu)$ any measure space

$\longrightarrow L^p(X, \mathcal{A}, \mu)$ is Banach $\forall p \in [1, \infty]$

Special case : $X = \mathbb{N} = \{0, 1, 2, \dots\}$

$\mathcal{A} = \mathcal{P}(\mathbb{N})$ "all subsets"

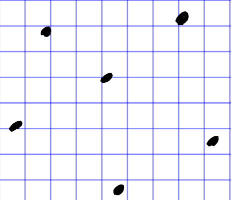
$\mu(A) = \#$ elements in A

μ ← counting measure

$$\ell^p := \mathcal{L}^p(X, \mathcal{A}, \mu) \quad \text{w/ choices above}$$

$$= \left\{ (x_n) : \begin{array}{ll} \sum |x_n|^p < \infty & \text{if } p < \infty \\ \sup\limits_{n \in \mathbb{N}} |x_n| & \text{if } p = \infty \end{array} \right\}$$

c) M any set , $(X, \|\cdot\|)$ Banach space



w/ these two ingredients we build another Banach space

$$\mathcal{B}(M, X) := \left\{ f : M \longrightarrow X \text{ s.t. } \sup_{m \in M} \|f(m)\|_X < \infty \right\}$$

↑ Banach space w/ operator norm

$$\|f\|_{\mathcal{B}(M,X)} := \sup_{m \in M} \|f(m)\|_X$$

reality check (exercise 3.1).

Prop. Every metric space can be isometrically embedded in a complete metric space. (*)

(idea : $\mathbb{Q} \longrightarrow \mathbb{R}$ . . . this can be done in general)

Def. (X_1, d_1) , (X_2, d_2) metric spaces

$\Phi : X_1 \longrightarrow X_2$ is called λ-Lipschitz if
$(\lambda \geq 0)$

$$d_2(\Phi(x_1), \Phi(x_1')) \leq \lambda\, d_1(x_1, x_1') \quad \forall x_1, x_1' \in X_1$$

$\Phi : X_1 \longrightarrow X_2$ is called an isometry if
(isometric embedding)

$$d_2(\Phi(x_1), \Phi(x_1')) = d(x_1, x_1') \quad \forall x_1, x_1' \in X_1.$$

(*): (a bit more precisely)

Given a metric space (M, d) there exists a $\underline{\text{complete}}$ metric space (M^*, d^*) and an isometry $\Phi : (M, d) \longrightarrow (M^*, d^*)$.

Proof.

take $X = \mathbb{R}$ w/ Euclidean norm

$B(M, \mathbb{R}) = M^*$, metric d^* induced by $\|\cdot\|_{Eucl}$

$$d^*(f^1, f^2) = \sup_{m \in M} |f^1(m) - f^2(m)|$$

we know (M^*, d^*) is complete (exercise above).

What is the isometry $\Phi : (M, d) \longrightarrow (M^*, d^*)$?

Fix $m^* \in M$ "base point"

$$(M, d) \xrightarrow{\Phi} (M^*, d^*)$$

$$m \longmapsto f_m$$

$$f_m(p) := d(m, p) - d(m^*, p)$$

(bounded? $|f_m(p)| = |d(m, p) - d(m^*, p)|$

$\leq d(m, m^*)$ uniform bound independent of p

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$f_m \in M^*$).

We must check Φ is an isometry, i.e.

$$\forall\, m_1, m_2 \in M \qquad d(m_1, m_2) = d^*(f_{m_1}, f_{m_2}) = \| f_{m_1} - f_{m_2} \|_{B(M,\mathbb{R})}$$

<u>Two inequalities:</u>

- $$\| f_{m_1} - f_{m_2} \|_{B(M,\mathbb{R})} = \sup_{p \in M} \Big| d(m_1, p) - \cancel{d(m^*, p)} - \big(d(m_2, p) - \cancel{d(m^*, p)}\big) \Big|$$
$$= \sup_{p \in M} \underbrace{| d(m_1, p) - d(m_2, p) |}_{\displaystyle \leq d(m_1, m_2)}$$
$$\overset{\text{tingle}}{\leq} d(m_1, m_2)$$

- $$d(m_1, m_2) = \Big| d(m_1, m_1) - d(m^*, m_1) - \big(d(m_2, m_1) - d(m^*, m_1)\big) \Big|$$
$$= | f_{m_1}(m_1) - f_{m_2}(m_1) | \leq \| f_{m_1} - f_{m_2} \|_{B(M,\mathbb{R})}.$$

$\sim\; \circ \;\sim$ ∎

<u>Remarks on norms:</u>

- if $(X, \|\cdot\|)$ is a normed space, then

$$\|\cdot\| : X \longrightarrow \mathbb{R} \quad \text{is 1-Lipschitz}$$
$(\Rightarrow$ continuous$)$

just use 2nd form of $\triangleleft$ triangle inequality

$$\big| \|x_1\| - \|x_2\| \big| \leq \|x_1 - x_2\|_X$$

- <u>equivalent norms</u>: vector space X — $\|\cdot\|$, — $\|\cdot\|'$

we call them equivalent if $\exists\, C > 0$ s.t.

$$C^{-1} \|x\| \leq \|x\|' \leq C \|x\| \qquad \forall x \in X$$

- if V vector space, $\dim V < \infty$ then any two norms are equivalent (Problem 1.1)

- an example of inequivalent norms:

$C^0([0,1])$ $\qquad \|f\| = \sup_{t\in[0,1]} |f(t)| \qquad L^\infty$

$$\|f\|' = \int_0^1 |f(t)|\,dt \qquad L^1$$

one inequality: $\|f\|' \leq \|f\|$

the converse ineq. is false:

$f_n(t) = t^n \quad n \geq 1 \qquad \|f_n\| = 1 \quad \forall n$

$$\|f_n\|' = \frac{1}{n+1} \xrightarrow[n\to\infty]{} 0$$