Functional Analysis I - L5 01/10/2020

Subspaces

$(X, \|\cdot\|)$ **normed space**, $Y \subset X$ subspace if it is closed under $+$ ("sum of vectors") and $\kappa \cdot \_$ ("multiplication by scalars").

Purely algebraic definition.

(1) in general, subspaces may or may not be closed.

1.1: $\dim Y < \infty \Longrightarrow Y$ is complete, hence closed.
$\qquad\qquad\qquad\qquad$ (#) $\qquad\qquad$ (*)

(*): (M, d) metric space, $Z \subset M$ subset then:

- $(Z, d|)$ complete $\Rightarrow Z \subset M$ closed
- if (M, d) complete, then
  $(Z, d|)$ complete $\Longleftrightarrow Z \subset M$ closed

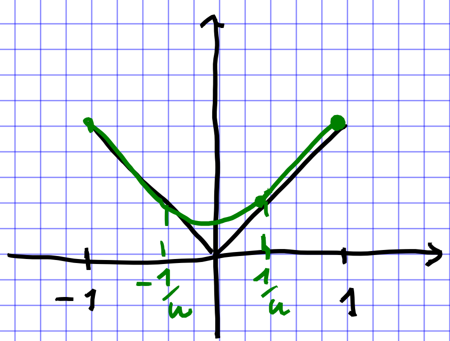
(#): for finite dim. linear spaces, any two norms are equivalent, so conclusion is reduced to the (known) fact that $(\mathbb{R}^n, \|\cdot\|_{Eucl})$ is complete.

1.2: Some examples of (vector) subspaces that are not closed:

(a) $C^1([-1,1]) \subset C^0([-1,1])$

here we work with norm $\|f\| = \max_{x \in [-1,1]} |f(x)|$

Claim: $C^1([-1,1])$ is NOT a closed subspace.



$$f_n(x) = \begin{cases} |x| & |x| \geq 1/n \\ \frac{n}{2}x^2 + \frac{1}{2n} & |x| \leq 1/n \end{cases}$$

$$f_n \xrightarrow[C^0]{} f(x) = |x|$$

⑥ $X = C^0([-1,1])$ w/ norm as above (Banach space).

$Y = \{$ restriction to $[-1,1]$ of $\mathbb{R}[x]$ polynomials $\}$

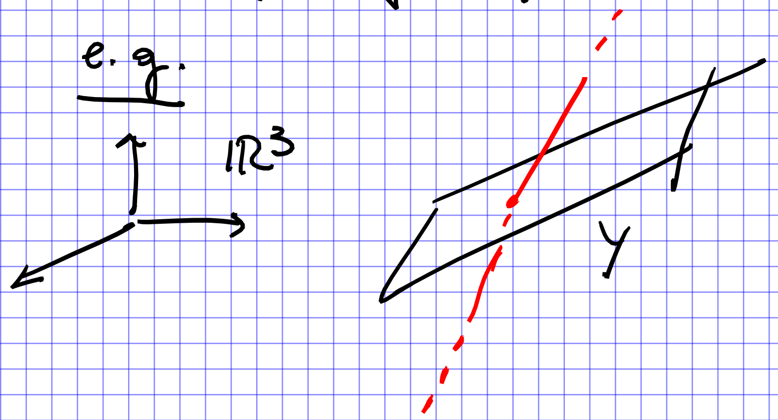
subspace, not closed

In fact, Stone-Weierstrass theorem implies

$$\overline{Y} = X.$$

② Key question on this matter is the question: given $(X, \|\cdot\|)$ and a linear subspace $Y \subset X$ can Y be "topologically complemented"?

e.g.



q. $\exists\, Z \subset X$
s.t. $X = Y \oplus Z$

this pb. is always solvable if $\dim X < \infty$ (complete the basis...) but to formulate and partially solve this pb. we need some background (stay tuned).

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Unit spheres

recall: (X, d) metric space, then TFAE:

1) is sequentially compact, i.e. $\forall\,(x_k)$ in X there exists $\underbrace{(x_{k(n)})}_{\text{subsequence}} \longrightarrow x_\infty \in X$;

2) $\cdot$ is compact (w.r.t. coverings), i.e. $\forall (U_i)_{i\in I}$ cover by open sets of X there exists a finite subcover

Also: we say that a subset $K \subset X$ is compact if and only if 1) or (equivalently!) 2) apply to (K, d).

<u>Rmk.</u> For $K \subset \mathbb{R}^n$ 1), 2) $\Longleftrightarrow$ K closed and bounded.

<u>Theorem</u>: $(X, \|\cdot\|)$ normed space. Then:

$$\dim X < \infty \iff \text{the unit sphere in } X$$

$$S := \{x \in X : \|x\| = 1\}$$

is compact

<u>Proof.</u>:

$\boxed{\Rightarrow}$ Fix a basis $\{u_1, \dots, u_n\}$ (if $\dim X = n$)

$$\mathbb{R}^n \longrightarrow X$$

$$(x_1, \dots, x_n) \xmapsto{\Phi} \sum_{i=1}^{n} x_i u_i \qquad \text{linear isomorphism}$$

Also: Φ is an isometry IF we place on $\mathbb{R}^n$ the pull-back norm

$$\|x\|^*_{\mathbb{R}^n} := \|\Phi(x)\|_X$$

Now:
- $S \subset X$ closed (pre-image of $\{1\}$ under the C^0 map $\|\cdot\| : X \to \mathbb{R}$)
- $S^* := \Phi^{-1}(S)$ closed by continuity of Φ
- S^* is closed in $(\mathbb{R}^n, \|\cdot\|_{Eucl})$ because $\|\cdot\|_{Eucl}$ is equivalent to $\|\cdot\|^*$ and equiv. norms induce the same topology

- S^* is bounded in $(\mathbb{R}^4, \|\cdot\|_{Eucl})$

again by

$$C^{-1}\|\cdot\|^* \leq \|\cdot\|_{Eucl} \leq C\|\cdot\|^*$$

- $\Longrightarrow$ S^* is compact in $(\mathbb{R}^4, \|\cdot\|_{Eucl})$
- S^* is compact in $(\mathbb{R}^4, \|\cdot\|^*)$ again by norm equivalent (same open sets $\Rightarrow$ same compact sets)
- $S = \Phi(S^*)$ compact in X because it is the continuous image of a compact set.

$\boxed{\Leftarrow}$ By contradiction:

assume $\dim X = \infty$ $\Longrightarrow$ $\exists$ sequence of points in the unit sphere S that has no converging subsequence

usual proof: let's pretend to be in a Hilbert space i.e. to have a scalar product $(\,,\,)$ on X.

$\exists$ (y_k) lin. independent family

$\Downarrow$ Gram-Schmidt orthonormalisation

$$(x_k) \text{ w/ } \begin{cases} \|x_k\| = 1 \\ (x_j, x_k) = 0 & \text{if } j \neq k \end{cases}$$

now: if we pick $j > k$

$$\begin{aligned} d(x_j, x_k)^2 &= \|x_j - x_k\|^2 \\ &= \|x_j\|^2 + \|x_k\|^2 - 2(x_j, x_k) \\ &= 1 + 1 - 0 = 2 \end{aligned}$$

$\Longrightarrow$ (x_k) has no Cauchy subsequence

$\Longrightarrow$ (x_k) has no converging subsequence.

<u>actual proof:</u> The $\boxed{\text{Riesz quasi-projection lemma}}$

Statement: let $(X, \|\cdot\|)$ normed vector space, $Y \subset X$ closed subspace $(Y \neq X)$. Then $\forall\, \epsilon > 0 \;\; \exists \;\; x = x(\epsilon)$ w/

1) $\|x\| = 1$

2) $d(x, Y) := \inf_{y \in Y} d(x, y) \geq 1 - \epsilon$.

<u>Pf</u> (Riesz lemma)

Y is closed (X∖Y open)

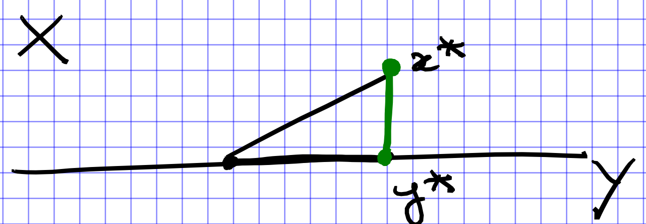
$Y \neq X \quad \Longrightarrow \quad \exists \;\; x^* \in X \setminus Y$

$$\text{set} \quad d := \inf_{y \in Y} d(x^*, y) > 0$$

$$\left(= d(x^*, Y) \right)$$

by def. of inf pick $y^* \in Y$ s.t.

$$d \leq d(x^*, y^*) \leq \frac{d}{1-\epsilon}$$



$$x = \frac{x^* - y^*}{\|x^* - y^*\|}$$

claim: this vector does the job

$\|x\| = 1$ <u>obvious</u>

$$d(x, Y) = \inf_{y \in Y} \|x - y\| = \inf_{y \in Y} \left\| \frac{x^* - y^*}{\|x^* - y^*\|} - y \right\|$$

$$= \inf_{y \in Y} \left\| \frac{x^* - \overbrace{y^* - \|x^* - y^*\| y}^{\in Y}}{\|x^* - y^*\|} \right\| \geq \frac{1}{\|x^* - y^*\|} \inf_{z \in Y} \|x^* - z\|$$

$$\geq \frac{\alpha}{\left(\frac{\alpha}{1-\epsilon}\right)} = 1-\epsilon \quad \text{what I wanted} \quad \blacksquare$$

Finally, let's complete the proof of non-compactness of S in Banach spaces:

$\dim X = \infty \rightsquigarrow (y_k)$ a sequence of lin. independent vectors

$$\rightsquigarrow \quad x_1 := \frac{y_1}{\|y_1\|}$$

inductively $Y_\ell = \langle y_1, \dots, y_\ell \rangle$

take $x_k \in Y_k \setminus Y_{k-1}$

s.t. $\|x_k\| = 1$

$d(x_k, Y_{k-1}) > 1/2$

for any $k \geq 2$

Then for any $k_2 > k_1$ have $d(x_{k_2}, x_{k_1}) > 1/2$

So (x_k) cannot have any converging subsequence. $\blacksquare$

Continuous Linear Maps

<u>Setup</u>: $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ normed spaces

$A: X \longrightarrow Y$ <u>linear</u>.

<u>Prop.</u> The following are equivalent:

i) A is continuous at $0 \in X$

ii) A is continuous at all $x_0 \in X$

iii) A is uniformly continuous (at all $x_0 \in X$)

iv) A is Lipschitz continuous (at all $x_0 \in X$)

v) $\underbrace{\sup_{\|x\|_X \leq 1} \|Ax\|_Y}_{=: \lambda} < \infty$

Remk from v) if $\lambda < \infty$ then (by scaling i.e. by linearity of A)

$$\|Ax\|_Y \leq \lambda \|x\|_X \,. \quad (*)$$

why? $\quad x \rightsquigarrow \bar{x} = \dfrac{x}{\|x\|_X} \rightsquigarrow \|\bar{x}\|_X = 1$

$\rightsquigarrow$ by def of λ $\quad \|A\bar{x}\|_Y \leq \sup\limits_{\|x\|_X \leq 1} \|Ax\|_Y = \lambda$

$\rightsquigarrow \quad \left\| A\left(\dfrac{x}{\|x\|_X}\right)\right\|_Y \leq \lambda$

$\rightsquigarrow$ linearity of A $\quad \|Ax\|_Y \leq \lambda \|x\|_X$.

Proof:

$$\begin{tikzcd} iv) \arrow[r, Rightarrow] & iii) \arrow[r, Rightarrow] & ii) \arrow[r, Rightarrow] & i) \arrow[dlll, Rightarrow] \\ v) \arrow[u, Rightarrow] & & & \end{tikzcd}$$

: recall $f: X \longrightarrow Y$ is λ-Lipschitz $(\lambda \geq 0)$ if

$$\|f(x_1) - f(x_2)\|_Y \leq \lambda \|x_1 - x_2\|_X \qquad \forall x_1, x_2 \in X$$

What we have to check:

$$\|Ax_1 - Ax_2\|_Y = \|A(x_1 - x_2)\|_Y$$

$$= \left\| A\left(\frac{x_1 - x_2}{\|x_1 - x_2\|_X}\right)\right\|_Y \cdot \|x_1 - x_2\|_X \leq \lambda \|x_1 - x_2\|_X$$

: by contradiction. Suppose $\sup\limits_{\|x\|_X \leq 1} \|Ax\|_Y = +\infty$

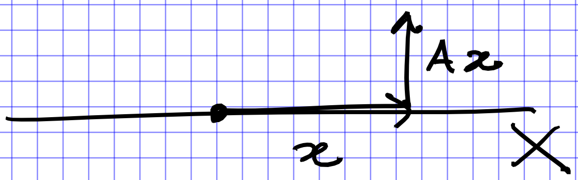
By def of sup, pick (x_n) w/ $\quad \|x_n\|_X \leq 1$

$(n \in \mathbb{N} = \{0, 1, 2, \dots\})$. $\quad \|Ax_n\|_Y \geq n$

set $z_n := \dfrac{x_n}{\|Ax_n\|_Y} \implies \begin{array}{l} z_n \longrightarrow 0 \\ Az_n \not\longrightarrow 0 \end{array} \implies A$ not continuous at $0 \in X$. ∎

Prop. : Finite-dimensional linear maps are automatically continuous.

Trick : take the graph-norm on X



$$\|x\|_* := \|x\|_X + \|Ax\|_Y$$

validity check : if A is linear then $\|\cdot\|_*$ is a norm on X.

If X is finite dimensional, then (equivalence of norms) $\exists\, C > 0$ w/ $(\forall x \in X)$

$$C^{-1} \|x\|_X \le \underbrace{\|x\|_* \le C \|x\|_X}$$

$$\|x\|_X + \|Ax\|_Y \le C \|x\|_X$$

$$\Longrightarrow \quad \|Ax\|_Y \le C \|x\|_X$$

$\Longrightarrow$ condition v) in Prop. above

$\Longrightarrow$ any linear $A: X \longrightarrow Y$ w/ X, Y finite dim'l spaces is continuous. ∎