Functional Analysis I - L6 5/10/2020

Continuous Linear Maps

$(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ normed spaces, then $A: X \to Y$ linear

criterion:
$$\underbrace{\sup_{\|x\|_X \le 1} \|Ax\|_Y}_{= \sup_{x \neq 0} \frac{\|Ax\|_Y}{\|x\|_X} = \sup_{\|x\|_X = 1} \|Ax\|_Y} < \infty$$

Example: a linear map that is not continuous.

$X = Y = C^0([0,1])$ as sets

$\|\cdot\|_X = \|\cdot\|_{L^1}$ $\qquad A = id$

$\|\cdot\|_Y = \|\cdot\|_{L^\infty}$

$$A \text{ is } C^0 \iff \sup_{\|f\|_{L^1} \le 1} \|f\|_{L^\infty} < \infty$$

$$\iff \boxed{\|f\|_{L^\infty} \le C \|f\|_{L^1} \quad \forall f \in C^0}$$

FALSE: for instance $f_n(t) = t^n$ $\quad \|f_n\|_{L^\infty} = 1$

$\|f_n\|_{L^1} = \frac{1}{n+1}$ ∎

Banach algebra of (continuous) linear maps:

Prop. $\left.\begin{array}{l}(X, \|\cdot\|_X) \text{ normed} \\ (Y, \|\cdot\|_Y) \text{ Banach}\end{array}\right\} \Rightarrow \underbrace{L(X,Y)}_{\text{continuous linear maps}}$ Banach space

w/ norm $\underbrace{\|A\|}_{\text{operator norm}} := \sup_{\|x\|_X \le 1} \|Ax\|_Y$.

BOUNDED lin. operators

Cor. $X = Y$ s.t. $(X, \|\cdot\|_X)$ Banach space, then

$$L(X) := L(X, X) \text{ is Banach.}$$

Proof. For a sequence (A_n) in $L(X,Y)$ being Cauchy means

$\forall \epsilon > 0 \quad \exists N = N(\epsilon)$ s.t. $\forall k, \ell \geq N$

$$(*) \qquad \| A_k - A_\ell \| \leq \epsilon \iff \sup_{\|x\|_X \leq 1} \|(A_k - A_\ell)x\|_Y \leq \epsilon$$

Note that $(*)$ has two implications:

(1) for any fixed $x \in X$ $(A_n x)$ is Cauchy in Y

$\left(\text{so, since } Y \text{ is Banach, } \exists \lim_{n\to\infty} A_n x = y =: Ax\right)$

(2) since $\big| \|A_k\| - \|A_\ell\| \big| \leq \|A_k - A_\ell\|$

then by $(*)$ we have that $a_n := \|A_n\|$ is a Cauchy sequence in $\mathbb{R}$.

Now:

- $A : X \longrightarrow Y$ is linear, since $A_n(x + x') = A_n x + A_n x'$

  take $n \to \infty$ $A(x+x') = Ax + Ax'$

- $A \in L(X,Y)$ i.e. it is a bounded linear operator

$$\|Ax\|_Y = \|\lim_{n\to\infty} A_n x\|_Y$$

$$= \lim_{n\to\infty} \underbrace{\|A_n x\|_Y}_{\leq \underbrace{\|A_n\| \cdot \|x\|_X}}$$

$$\leq \underbrace{\left(\lim_{n\to\infty} \|A_n\|\right)}_{\text{using (2)}} \cdot \|x\|_X$$

$$= \underset{\in \mathbb{R}}{\lambda} \cdot \|x\|_X$$

$$\implies \|A\| \leq \lambda$$

- $A_n \longrightarrow A$ in the operator norm

recall: given $\epsilon > 0$ $\exists N = N(\epsilon)$ so that $\forall n, \ell \geq N$

$$\sup_{\|x\|_X \leq 1} \| (A_\ell - A_n) x \| \leq \epsilon$$

Then $A_\ell x \xrightarrow{\ell \to \infty} Ax \Rightarrow (A_\ell - A_n)x \xrightarrow{\ell \to \infty} (A - A_n)x$

$$\Longrightarrow \quad \| (A_\ell - A_n) x \|_Y \xrightarrow{\ell \to \infty} \| (A - A_n) x \|_Y$$

hence $\| (A - A_n) x \|_Y = \lim_{\ell \to \infty} \| (A_\ell - A_n) x \|_Y$

$$\leq \underbrace{\left(\limsup_{\ell \to \infty} \| A_\ell - A_n \| \right)}_{\leq \epsilon \quad \text{if} \quad n \geq N = N(\epsilon)} \|x\|_X$$

$$\Longrightarrow \quad \sup_{\|x\|_X \leq 1} \frac{\| (A - A_n) x \|_Y}{\|x\|_X} \leq \epsilon \quad \text{if} \quad n \geq N(\epsilon)$$

$$\Longrightarrow \quad A_n \xrightarrow{n \to \infty} A \quad \text{in } L(X, Y) \quad \blacksquare$$

To proceed, and discuss about "composition of operators" it is first convenient to talk about <u>products</u> of Banach spaces.

Products of Banach spaces

Let $(X_i, \|\cdot\|_i)$ $1 \leq i \leq n$ <u>normed</u> spaces.

- on the product $X := \prod_{i=1}^{n} X_i$ one can place norms

$$\| x \|_{p,X} = \left(\sum_{i=1}^{n} \| x_i \|_{X_i}^p \right)^{1/p}$$

if $x = (x_1, x_2, \dots, x_n)$ and $p \in [1, \infty]$

Special cases:

$$\| x \|_{1,X} = \sum_{i=1}^{n} \| x_i \|_{X_i}, \quad \| x \|_{\infty, X} = \max_{i \in \{1, \dots, n\}} \| x_i \|_{X_i}$$

Given any $p, q \in [1, \infty]$ we have that $\|\cdot\|_{p,X}$ is equivalent to $\|\cdot\|_{q,X}$ as follows from the finite-dim result in $\mathbb{R}^n$:

$$C_{p,q}^{-1} \|x\|_{p,\mathbb{R}^n} \leq \|x\|_{q,\mathbb{R}^n} \leq C_{p,q} \|x\|_{p,\mathbb{R}^n}$$

one can determine $C_{p,q}$ explicitly

- the projection maps are continuous

$$\pi_i : X \longrightarrow X_i$$
$$x = (x_1, \dots, x_n) \longmapsto x_i$$

$$\| \pi_i x \|_{X_i} = \| x_i \|_{X_i} \leq \| x \|_{p,X} \quad \forall\, p \in [1, \infty]$$

- if $\forall i$ $(X_i, \|\cdot\|_{X_i})$ is <u>complete</u> (i.e. it is a Banach space) then so it is the product $\left(X = \prod_{i=1}^n X_i, \|\cdot\|_{p,X}\right)$ $p \in [1, \infty]$

Same proof as $(\mathbb{R}^n, \|\cdot\|_{Eucl})$ is complete, given that $(\mathbb{R}, \|\cdot\|_{Eucl})$ is complete.

Composition of bounded linear operators

<u>Prop.</u> : $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, $(Z, \|\cdot\|_Z)$ <u>normed spaces</u>

(1) $A \in L(X, Y)$, $B \in L(Y, Z) \Longrightarrow BA \in L(X, Z)$

and $\quad \|BA\|_{L(X,Z)} \leq \|A\|_{L(X,Y)} \|B\|_{L(Y,Z)}$

(2) the composition operation

$$\underbrace{L(X, Y) \times L(Y, Z)}_{\text{product of normed spaces}} \longrightarrow L(X, Z)$$

is continuous.

Proof: (1)
$$\sup_{x\neq 0} \frac{\| BA(x)\|_Z}{\|x\|_X} = \sup_{x\neq 0} \frac{\|BA(x)\|_Z}{\|Ax\|_Y}\frac{\|Ax\|_Y}{\|x\|_X}$$

WLOG can assume $x \notin \mathrm{Ker}(A)$

$$\leq \left(\sup_{x\neq 0}\frac{\|Ax\|_Y}{\|x\|_X}\right)\left(\sup_{x\neq 0}\frac{\|BAx\|_Z}{\|Ax\|_Y}\right)$$

$$\leq \left(\sup_{x\neq 0}\frac{\|Ax\|_Y}{\|x\|_X}\right)\left(\sup_{y\neq 0}\frac{\|By\|_Z}{\|y\|_Y}\right)$$

$$= \|A\|\cdot\|B\| \implies \|BA\| \leq \|A\|\|B\|.$$

(2) algebraic trick:
$$BA - B_0A_0 = (B-B_0)A + B_0(A-A_0)$$

take operator norms
$$\|BA - B_0A_0\| \leq \|(B-B_0)A\| + \|B_0(A-A_0)\|$$
$$\underset{\text{part (1)}}{\leq} \|B-B_0\|\,\|A\| + \|B_0\|\|A-A_0\|$$

now if (A,B) close to (A_0, B_0) in $L(X,Y)\times L(Y,Z)$ then both $\|A-A_0\|$, $\|B-B_0\|$ are "small", thus by previous ineq. $\|BA - B_0A_0\|$ is small as well. ∎

Motivational comment: given $A \in L(X)$ we made sense of
$$A^2, A^3, \dots, A^k, \dots$$
and note $\|A^k\| \leq \|A\|^k$

but how about $\sum_{k=0}^{\infty} A^k$ or maybe $e^A \overset{\text{moral equality/hope}}{=} \sum_{k=0}^{\infty}\frac{A^k}{k!}$

Need a good convergence criterion.

Prop. (normal convergence criterion) Take $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ Banach, consider $L(X,Y)$ bounded linear op.
Let for $j \in \mathbb{N}$ $A_j \in L(X,Y)$ s.t. $\sum_{j=0}^{\infty} \|A_j\|_{L(X,Y)} < \infty$

Then there exists $\underbrace{\sum_{j=0}^{\infty} A_j}_{\in L(X,Y)} = \lim_{n\to\infty} \sum_{j=0}^{n} A_j$.

Proof. Since $L(X,Y)$ is Banach (because Y is Banach) so it's enough to check that the partial sums $S_n := \sum_{j=0}^{n} A_j$ are Cauchy, i.e. $\forall \epsilon > 0$ $\exists N = N(\epsilon)$ w/ $\forall n, \ell \geq N$ have $\|S_n - S_\ell\|_{L(X,Y)} < \epsilon$

But $\|S_n - S_\ell\|_{L(X,Y)} \equiv \left\| \sum_{j=\ell+1}^{n} A_j \right\|_{L(X,Y)}$

$$\leq \sum_{j=\ell+1}^{n} \|A_j\|_{L(X,Y)} \leq \underbrace{\sum_{j\geq \ell+1} \|A_j\|}_{\text{tail of a converging series: as small as we wish if we take } \ell+1 \text{ large enough}}$$

■

Two important examples:

(1) (towards functional calculus) given $A \in L(X)$, w/ X Banach

Claim: $\exists \underbrace{\exp(A)}_{\in L(X)} := \sum_{k=0}^{\infty} \frac{A^k}{k!}$

why? apply criterion above

$$\left\| \frac{A^k}{k!} \right\| = \frac{1}{k!} \|A^k\| \leq \underbrace{\frac{1}{k!} \|A\|^k}_{\text{summable!}}$$

Set $a := \|A\|$ $\sum_{n=0}^{\infty} \frac{a^k}{k!} = e^a$

hence criterion applies and $\|e^A\| \leq e^{\|A\|}$ ■

② **Neumann series** : X Banach, $A \in L(X)$ w/ $\|A\| < 1$

Claim : $\exists \underbrace{\sum_{k=0}^{\infty} A^k}_{\in L(X)}$ (just apply criterion to a geometric series)

Claim : $\sum_{k=0}^{\infty} A^k$ is a 2-sided inverse of $\underbrace{\mathbb{1} - A}_{\in L(X)}$.

$$S_n := \sum_{k=0}^{n} A^k$$

$$\boxed{(\mathbb{1} - A) S_n = \mathbb{1} - A^{n+1}}$$

now, just pass to the limit in ↑.

$$\text{LHS} \xrightarrow{n\to\infty} (\mathbb{1} - A) \sum_{k=0}^{\infty} A^k$$

$$\text{RHS} \xrightarrow{n\to\infty} \mathbb{1}$$

because $\|A^{n+1}\| \leq \|A\|^{n+1} \xrightarrow[\|A\|<1]{n\to\infty} 0$

Put things together $(\mathbb{1} - A) \underbrace{\sum_{k=0}^{\infty} A^k}_{\in L(X)} = \mathbb{1}$

Similarly $\left(\sum_{k=0}^{\infty} A^k\right)(\mathbb{1} - A) = \mathbb{1}$.