Functional Analysis I – L7   8/10/2020

Spectral Radius

$(X, \|\cdot\|_X)$ normed space, $A \in \underbrace{L(X)}_{\text{bounded linear operators } X \to X}$

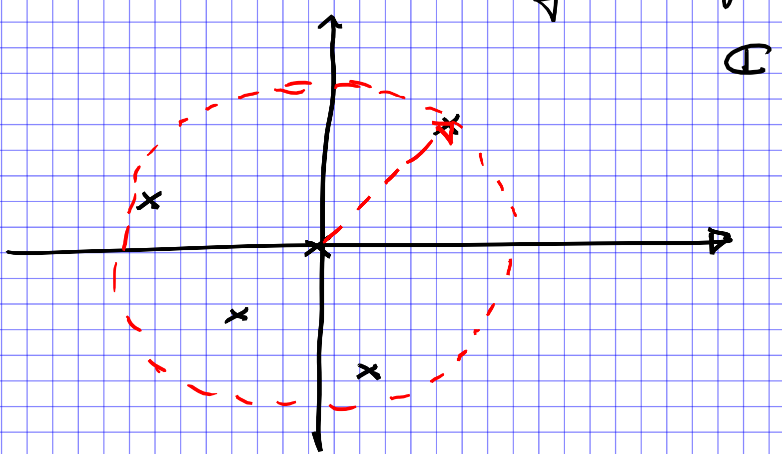
<u>Thm</u>: there exists $\lim_{n\to\infty} \|A^n\|_{L(X)}^{1/n} = r_A$ ← spectral radius of A

<u>Rmk. 1</u>: we'll prove that in fact $\lim_{n\to\infty} \|A^n\|_{L(X)}^{1/n} = \inf_{n\in\mathbb{N}} \|A^n\|_{L(X)}^{1/n}$.

<u>Rmk. 2</u>: $0 \le r_A \le \|A\| \quad \left(\|A^n\|_{L(X)} \le \|A\|_{L(X)}^n \right)$

<u>Rmk. 3</u>: say that $\lambda \in \mathbb{K}$ $(\mathbb{K} = \mathbb{R}, \mathbb{C})$ is an <u>eigenvalue</u> of A if $\exists\, x_\lambda \neq 0$ in X $Ax_\lambda = \lambda x_\lambda$. Thus $A^n x_\lambda = \lambda^n x_\lambda \implies r_A \ge |\lambda|$.

For matrices r_A is the norm of the largest eigenvalue



<u>Cor.</u> In the setting above, if $r_A < 1$ then
$$\exists \sum_{k=0}^{\infty} A^k \in L(X), \text{ thus } \mathbb{1} - A \text{ is invertible}$$
(through its Neumann series).

Proof of Cor. given thm.: recall "ratio criterion" for convergence of series i.e. given a sequence (a_n) of real / complex numbers

if $\limsup_n \sqrt[n]{|a_n|} < 1$ then $\sum_{n=0}^{\infty} a_n$ is convergent.

Now, apply this criterion $a_n := \| A^n \|_{L(X)}$:

$$r_A < 1 \implies \sum_{n=0}^{\infty} \| A^n \| < \infty \xRightarrow[\text{normal convergence}]{} \sum_{n=0}^{\infty} A^n \in L(X).$$

∎

Proof (spectral radius thm.) given $\epsilon > 0$ choose $k = k(\epsilon)$ w/

$$\| A^k \|_{L(X)}^{1/k} \leq \inf_n \| A^n \|_{L(X)}^{1/n} + \epsilon$$

We can now consider positive integers modulo k, i.e we write

$$\boxed{n = k\ell + m} \quad \text{for } 0 \leq m < k$$

this way we have two well-defined sequences $\ell(n)$, $m(n)$.

divide by nk

$$(*) \quad \boxed{\frac{1}{k} = \frac{\ell}{n} + \frac{1}{k}\frac{m}{n}} \quad \text{where } \ell = \ell(n), \ m = m(n)$$

Thus

$$\| A^n \|_{L(X)}^{1/n} = \| A^{k\ell + m} \|^{1/n} \leq \underbrace{\| A^k \|^{\ell/n}}_{\downarrow \ \| A^k \|^{1/k}} \cdot \underbrace{\| A \|^{m/n}}_{\downarrow \ 1} \quad (n \to \infty)$$

but now $\dfrac{m(n)}{n} \xrightarrow[n\to\infty]{} 0$

and by $(*)$ $\dfrac{\ell(n)}{n} \xrightarrow[n\to\infty]{} \dfrac{1}{k}$

Therefore

$$\limsup_{n\to\infty} \| A^n \|^{1/n} \leq \| A^k \|^{1/k} \leq \inf_n \| A^n \|^{1/n} + \epsilon$$

by definition of k

True for all ϵ $\rightsquigarrow$ $\limsup_{n\to\infty} \| A^n \|^{1/n} \leq \inf_n \| A^n \|^{1/n} \leq \liminf_{n\to\infty} \| A^n \|^{1/n}$

$$\boxed{\text{Conclusion: } \exists \lim_{n\to\infty} \| A^n \|^{1/n} = \inf_n \| A^n \|^{1/n}}$$

∎

LINEAR GROUP IN A BANACH SPACE:

$(X, \|\cdot\|_X)$ Banach space, $L(X)$ bounded lin. operator

is itself a Banach space

$$GL(X) = \{ A \in L(X) : A \text{ invertible } w/\ A^{-1} \in L(X) \}$$

will come for free by the open mapping theorem

Prop. $GL(X) \subset L(X)$ is open.

Proof: claim: the whole metric ball — pick $A_0 \in GL(X)$

$$\Omega = \{ A \in L(X) : \|A - A_0\|_{L(X)} < \|A_0^{-1}\|_{L(X)}^{-1} \} \subset GL(X)$$

$$\Big(\mathbb{1} = A_0 A_0^{-1} \implies 1 = \|A_0 A_0^{-1}\| \leq \|A_0\| \|A_0^{-1}\|$$

$$\implies \|A_0^{-1}\|_{L(X)} > 0 \Big)$$

$$A = \underbrace{A_0}_{\in GL(X) \text{ by assumption}} \underbrace{\left(\mathbb{1} + A_0^{-1} (A - A_0) \right)}_{\text{it's enough to show that } \in GL(X)}$$

Using Neumann series, we simply need to check that $\|A_0^{-1}(A - A_0)\|_{L(X)} < 1$

But if $A \in \Omega$ then

$$\|A_0^{-1}(A - A_0)\|_{L(X)} \leq \|A_0^{-1}\|_{L(X)} \underbrace{\|A - A_0\|}_{< \|A_0^{-1}\|_{L(X)}^{-1}}$$

$$< 1$$

Thus the claim is true and so $GL(X) \subset L(X)$ is open. ∎

Quotients

X vector space over k, $Y \subset X$ <u>subspace</u> $(Y \neq X)$

Lin. Algebra level: $X/Y = \{ [x] : x \in X \}$

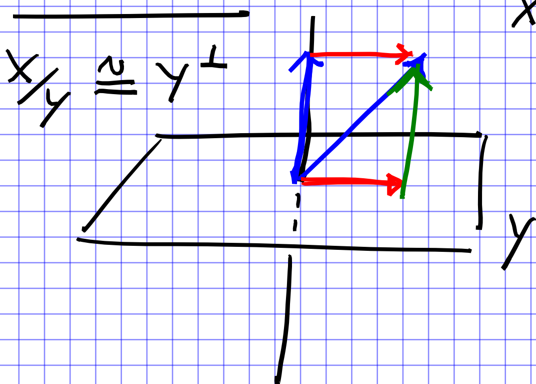
where the equivalence relation is $x_1 \sim x_2 \iff x_1 - x_2 \in Y$

<u>Facts</u>: X/Y is a vector space over k,

$$\begin{cases} \alpha [x] := [\alpha x] \\ [x_1] + [x_2] := [x_1 + x_2] \end{cases}$$

<u>Intuitive</u>:

$X = \mathbb{R}^3$



this picture, which is always good to keep in mind, is still rigorously true for Hilbert space (see next lectures)

<u>Thm</u>: 1) $(X, \|\cdot\|_X)$ normed space, $Y \subset X$ <u>closed</u> subspace

then $(X/Y, \|\cdot\|_{X/Y})$ is normed w/

$$\| [x] \|_{X/Y} = \inf_{y \in Y} \|x - y\|_X$$

2) $\pi : X \longrightarrow X/Y$ is continuous, w/ operator norm

$x \longmapsto \pi(x) = [x] \qquad \|\pi\|_{L(X, X/Y)} = 1$

3) if $(X, \|\cdot\|_X)$ Banach space, then $(X/Y, \|\cdot\|_{X/Y})$ <u>Banach</u>.

<u>Rmk.</u> need some topological hp. on Y. Note that if Y is dense (e.g. $C^0([0,1]) \subset L^1([0,1])$) then $\forall x \in X$ $\inf_{y \in Y} \|x - y\| = 0$.

$\Longrightarrow$ no way one can even define a norm this way!

Proof:

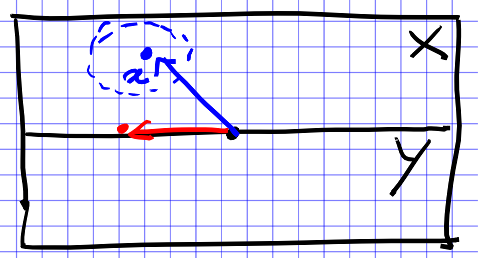
(1) • $\| [x] \|_{X/Y} = 0 \iff [x] = 0$ in X/Y

$\boxed{\Leftarrow}$ $[x]=0$ in $X/Y \Rightarrow x \in Y \Rightarrow \inf_{y\in Y} \|x-y\| \leq \|x-x\| = 0$

$\boxed{\Rightarrow}$ if $[x] \neq 0$ in X/Y then $x \in X \setminus Y$, but $X \setminus Y$ is open

$\Rightarrow \exists r>0 \quad B_r(x) \subset X \setminus Y$

$\Rightarrow \inf_{y\in Y} \|x-y\| \geq r > 0$.



• $\| [\alpha x] \|_{X/Y} = |\alpha| \, \|[x]\|_{X/Y}$

if (y_k) seq. in Y s.t. $\|x-y_k\| \longrightarrow \inf_{y\in Y} \|x-y\|$

then (αy_k) seq in Y s.t. $\underbrace{\|\alpha x - \alpha y_k\|}_{|\alpha| \, \|x - y_k\|} \longrightarrow \inf_{z\in Y} \|\alpha x - z\|$

• $\| [x_1+x_2] \|_{X/Y} \leq \| [x_1] \|_{X/Y} + \| [x_2] \|_{X/Y}$

Given $\epsilon > 0$ pick $\begin{cases} y_1 \in Y \\ y_2 \in Y \end{cases}$ w/ $\begin{array}{l} \|x_1 - y_1\|_X < \|[x_1]\|_{X/Y} + \epsilon/2 \\ \|x_2 - y_2\|_X < \|[x_2]\|_{X/Y} + \epsilon/2 \end{array}$

$$\| [x_1+x_2] \|_{X/Y} = \inf_{y\in Y} \|(x_1+x_2) - y\|_X$$

take $y = y_1 + y_2$ as a legitimate choice

$$\leq \|(x_1+x_2) - (y_1+y_2)\|_X$$

$$= \|(x_1 - y_1) + (x_2 - y_2)\|_X$$

$$\leq \|x_1 - y_1\|_X + \|x_2 - y_2\|_X$$

$$\leq \|[x_1]\|_{X/Y} + \|[x_2]\|_{X/2} + \epsilon$$

Let $\epsilon \to 0$ to get $\| [x_1+x_2] \|_{X/Y} \leq \| [x_1] \|_{X/Y} + \| [x_2] \|_{X/Y}$.

② $\| \pi(x) \|_{X/Y} = \| [x] \|_{X/Y} = \inf_{y \in Y} \|x - y\| \leq \|x\|$

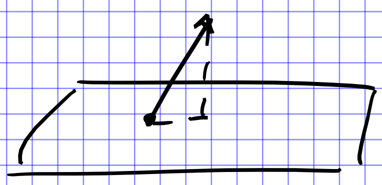
$\Longrightarrow \quad \dfrac{\| \pi(x) \|_{X/Y}}{\|x\|_X} \leq 1 \quad \Longrightarrow \quad \| \pi \|_{L(X, X/Y)} \leq 1 .$

For the converse inequality, use Riesz Lemma

if $Y \subsetneq X$ closed given $\epsilon > 0 \ \exists \ x = x_\epsilon$ w/ $\|x\| = 1$

$\underbrace{d(x, Y)}_{= \inf_{y \in Y} \|x - y\| = \| [x] \|_{X/Y}} \geq 1 - \epsilon$

thus



$\Longrightarrow \quad \| [x] \|_{X/Y} \geq 1 - \epsilon$

$\Longrightarrow \quad \| \pi \|_{L(X, X/Y)} \geq 1 - \epsilon$ ⟵ true for any $\epsilon > 0$

conclusion $\| \pi \|_{L(X, X/Y)} \geq 1 \ \underset{1^{st} \text{ part}}{\leadsto} \ \| \pi \|_{L(X, X/Y)} = 1 .$

③ want $(X/Y, \| \cdot \|_{X/Y})$ is complete (if so is $(X, \| \cdot \|_X)$)

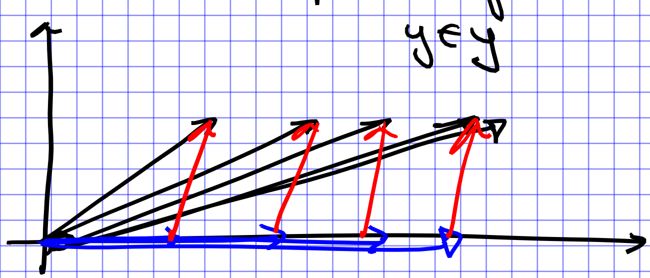
$([x_k])$ Cauchy, WLOG by taking a subsequence (no rename)

$\| [x_k] - [x_{k-1}] \|_{X/Y} < 2^{-k} \qquad k > 1$

pick $y_1 = 0$ and for $k > 1$ take y_k an "almost-minimizer"
$z_1 = x_1$

for $\inf_{y \in Y} \|x_k - y\|$

in part. we require



that the "associated vertical sequence" (red arrows)

$$z_k := x_k - y_k$$

satisfies

$$\| z_k - z_{k-1} \|_X \leq \| [z_k] - [z_{k-1}] \|_{X/Y} + 2^{-k}$$

$$\equiv \| [x_k] - [x_{k-1}] \|_{X/Y} + 2^{-k}$$

$$< \underbrace{2^{-k}} + 2^{-k} = 2^{1-k}$$

thus (z_n) is Cauchy in $\underbrace{(X, \|\cdot\|_X)}$

but $\cdot$ is Banach, so $z_k \longrightarrow z \in X$

thus, by continuity of π (part 2) of the proof)

$$\begin{array}{ccc} \pi(z_k) & \longrightarrow & \pi(z) \\ \| & & \\ [z_k] & = & [x_k] \end{array}$$

so $([x_k])$ is convergent in X/Y, limit point $[z] = \pi(z)$.

To conclude, we use:

$$\left.\begin{array}{c} \text{Cauchy} \\ \oplus \\ \text{one converging subsequence} \end{array}\right\} \Rightarrow \underline{\text{convergent}}.$$

$\blacksquare$