Functional Analysis I - L8 12/10/2020

Hilbert Spaces

X vector space / $\mathbb{C}$ (special case: / $\mathbb{R}$)

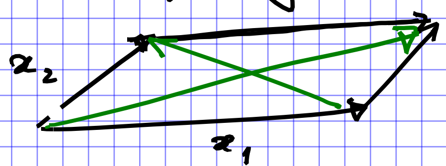
Def: $(\, , \,) : X \times X \longrightarrow \mathbb{C}$ hermitian product if

1) $(x, x) \geq 0$, $(x,x) = 0 \iff x = 0$ $(\forall x \in X)$
2) $(y, x) = \overline{(x,y)}$ $(\forall x, y \in X)$
3) $(\alpha x_1 + \beta x_2, y) = \alpha (x_1, y) + \beta (x_2, y)$ $(\forall x_1, x_2, y \in X$, $\forall \alpha, \beta \in \mathbb{C})$

Def: A pair $(X, (\, , \,))$ is called a Hilbert space if w.r.t. $\boxed{\|x\| = \sqrt{(x,x)}}$ the space is complete.

morally: Hilbert means "complete space w. hermitian product".

In part. Hilbert spaces are a subclass of Banach spaces, and they are characterized by the validity of the "parallelogram identity"



$$\|x_1 + x_2\|^2 + \|x_1 - x_2\|^2 = 2\|x_1\|^2 + 2\|x_2\|^2$$

Rmk. to check that $\boxed{}$ defines a norm, the only "non-trivial" thing is the triangle inequality

$$\|x_1 + x_2\| \leq \|x_1\| + \|x_2\|$$

For that, we need Cauchy-Schwartz, i.e.

$$(*) \qquad |(x,y)| \leq \|x\| \, \|y\| \qquad (\forall x, y \in X)$$

Proof: • if $\|x\| = 0$ or $\|y\| = 0$ nothing to prove

• else scale them so that $\|x\| = 1$, $\|y\| = 1$, must check

$$|(x,y)| \leq 1.$$

$$1 = \|y\|^2 = \| tx + (y-tx) \|^2 \qquad (t \in \mathbb{C})$$

$$= |t|^2 \|x\|^2 + \|y - tx\|^2 + 2t \,\mathrm{Re}\,(x, y-tx)$$

<u>choose</u> $t = \overline{(x,y)}$ so that $(x, y - tx) = 0$

$$1 \geq |t|^2 \|x\|^2 = |t|^2 = |\overline{(x,y)}|^2 = |(x,y)|^2. \quad \blacksquare$$

<u>rmk.</u> equality $\iff$ $y = sx$ $\quad s \in \mathbb{C}$.

<u>Examples</u>:

1) $\mathbb{C}^n$ w/ $\quad w \cdot z := \sum_{i=1}^{n} w_i \overline{z_i}$

2) $L^2((0,1);\mathbb{C}) \qquad (f,g)_{L^2} := \int_0^1 f(t)\overline{g(t)}\,dt$

3) (special case of 2))

$\ell^2_{\mathbb{C}} \qquad (x,y) = \sum_{k=0}^{\infty} x_k \overline{y_k}$.

<u>Orthogonality</u>

Setting as above. Take $Y \subset X$ <u>subspace</u>.

$$Y^\perp = \{ x \in X : (x,y) = 0 \ \forall y \in Y \}$$

<u>Lemma</u>: given any subspace Y, then $Y^\perp$ is always a <u>closed</u> subspace.

<u>Proof</u>: • given $y \in X$ (in part. $y \in Y$)

$$\ell_y : X \longrightarrow \mathbb{C} \qquad (\text{a } \mathbb{C}\text{-linear map})$$
$$z \longmapsto (z,y)$$

this defines a <u>bounded</u> operator by the Cauchy-Schwarz inequality

$$|\ell_y(z)| = |(z,y)| \leq \underbrace{\|y\|}_{C} \|z\|$$

$$|\ell_y(z)| \leq C\|z\|.$$

Hence (by continuity) $\ker(\ell_y) \subset X$ <u>closed</u> subspace

$$Y^\perp = \bigcap_{y \in Y} \ker(\ell_y) \quad \underline{\text{closed}} \quad \blacksquare$$

Basics about orthogonals.

a) $Y_1 \subset Y_2 \implies Y_1^\perp \supset Y_2^\perp$

b) $Y = \underline{0} \implies Y^\perp = X$

c) $\boxed{Y^\perp = \overline{Y}^\perp}$

d) $\overline{Y} = X \implies Y^\perp = 0$ (use c))

$$Y \subset \overline{Y} \overset{a)}{\implies} Y^\perp \supset \overline{Y}^\perp$$

Let's check the converse inclusion: $x_0 \in Y^\perp$ $\left(\overset{?}{\implies} x_0 \in \overline{Y}^\perp\right)$

<u>Know</u> $(x_0, y) = 0 \quad \forall y \in Y$

$\bar{y} \in \overline{Y}$ i.e. $\exists\, (y_k)$ $\quad y_k \to \bar{y}$

$$(x_0, y_k) = 0 \quad \text{(hp.)}$$

$$(x_0, \bar{y}) = \underbrace{(x_0, y_k)}_{=0} + \underbrace{(x_0, \bar{y} - y_k)}$$

$$|\cdot| \le \|x_0\| \underbrace{\|\bar{y} - y_k\|}_{\substack{\to 0 \\ k \to \infty}}$$

Cauchy-Schwartz

$\implies x_0 \in \overline{Y}^\perp.$ ∎

<u>Prop.</u> (projections on a closed subspace)

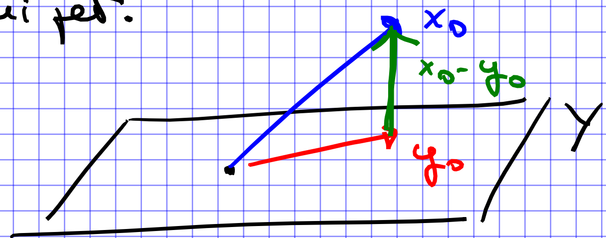
$(X, (\cdot,\cdot))$ Hilbert space $/ \mathbb{C}$, $Y \subset X$ <u>closed</u> subspace.

- Given $x_0 \in X$ $\exists!$ $y_0 = \pi_Y(x_0) \in Y$ s.t.

$$(x_0 - y_0, y) = 0 \quad \forall y \in Y$$

- Furthermore, $d(x_0, Y) = \inf_{y \in Y} d(x_0, y) = d(x_0, y_0)$

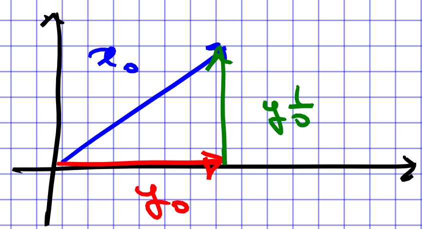
and y_0 is the <u>only</u> minimizer.



Applications and corollaries:

① $X = Y \oplus Y^\perp$ (if Y closed)

$x_0 \quad y_0 \quad y_0^\perp$



$$\|x_0\|^2 = \|y_0\|^2 + \|y_0^\perp\|^2$$

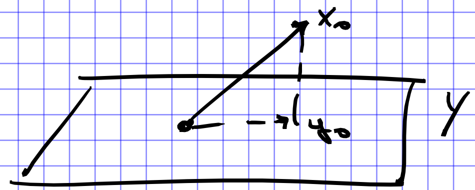
$$Id_X = \pi_Y \oplus \pi_{Y^\perp}$$

if $0 \subsetneq Y \subsetneq X$ then $\begin{cases} \|\pi_Y\|_{L(X)} = 1 \\ \|\pi_{Y^\perp}\|_{L(X)} = 1 \end{cases}$

② (exercise 5.1) if $Y \subset X$ closed

$$X/Y \underset{\text{isometric}}{\cong} Y^\perp$$

Proof (proposition)



Part I: existence of a minimizer

everything trivial if $x_0 \in Y$ (just take $y_0 = x_0 \ldots$)

Else suppose $x_0 \notin Y$ (i.e. $x_0 \in X \setminus Y$)

$$\underbrace{0 < \underline{d}} = d(x_0, Y) = \inf_{y \in Y} d(x_0, y)$$

pick (y_k) minimizing sequence i.e. $d(x_0, y_k) \longrightarrow \underline{d}$

Claim: $\underbrace{(y_k) \text{ is Cauchy}}$, thus $y_k \longrightarrow y_\infty \in Y$

w/ $d(x_0, y_\infty) = \underline{d}$

must use the parallelogram identity

$\begin{cases} a = x_0 - y_k \\ b = x_0 - y_\ell \end{cases}$

$$2\left(\|x_0 - y_k\|^2 + \|x_0 - y_\ell\|^2\right) = \|y_k - y_\ell\|^2 + 4\left\|x_0 - \frac{y_k + y_\ell}{2}\right\|^2$$

$$\|y_k - y_\ell\|^2 = 2\left(\|x_0 - y_k\|^2 + \|x_0 - y_\ell\|^2\right) - 4\left\|x_0 - \frac{y_k + y_\ell}{2}\right\|^2$$

$$\|x_0 - y_k\| = \underline{d} + o_k(1)$$

$$\|x_0 - y_\ell\| = \underline{d} + o_\ell(1)$$

$$\leq \cancel{2\underline{d}^2} + o_k(1) + \cancel{2\underline{d}^2} + o_\ell(1) - \cancel{4\underline{d}^2} = o_k(1) + o_\ell(1)$$

Part II (geometric characterization): for a vector $y_0 \in Y$ TFAE:

i) $\quad d(x_0, y_0) = \underline{d} = \inf_{y \in Y} d(x_0, Y)$

ii) $\quad (x_0 - y_0, y) = 0 \quad \forall y \in Y \quad (\iff x_0 - y_0 \perp Y)$.

i) $\Longrightarrow$ ii) given any $y \in Y$

$$p(t) := \|(x_0 - y_0) + t y\|^2$$

has a minimum at $t = 0$

$$\left[\frac{d}{dt}\right]_{t=0} p(t) = 0$$

(Euler–Lagrange equation) Compute it:

$$\left[\frac{d}{dt}\right]_{t=0} = \boxed{2\,\mathrm{Re}\,(x_0 - y_0, y) = 0}$$

similarly argue for iy in place of y $\qquad \forall y \in Y$

$$\boxed{2\,\mathrm{Im}\,(x_0 - y_0, y) = 0}$$

Conclusion: $\quad (x_0 - y_0, y) = 0 \quad \forall y \in Y$

ii) $\Longrightarrow$ i) $\quad \underbrace{p(t)}_{\text{as above}} = \|x_0 - y_0\|^2 + t^2\|y\|^2$

orthogonality hp.

$$d^2(x_0, y_0 - ty) = \|x_0 - y_0\|^2 + \underbrace{t^2\|y\|^2}_{\geq 0}$$

$$\geq \|x_0 - y_0\|^2 \quad \Longrightarrow \quad y_0 \text{ is a minimizer}$$

Part III : uniqueness of the minimizer

take y_0, y_0^* two minimizers $\xrightarrow{\text{part II}}$ $\begin{cases} (x_0 - y_0, y) = 0 & \forall y \in Y \\ (x_0 - y_0^*, y) = 0 & // \end{cases}$

set $y := y_0 - y_0^*$

$$\|(x_0 - y_0) + y\|^2 = \|x_0 - y_0^*\|^2 = \cancel{d^2}$$

$$= \|x_0 - y_0\|^2 + \|y\|^2 = \cancel{d^2} + \|y\|^2$$

$\Longrightarrow \|y\|^2 = 0 \Longrightarrow y = 0.$ ■

In other words $y_0 = y_0^*$.

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Double normals : given $Y \subset X$ subspace

$$Y^{\perp\perp} := (Y^\perp)^\perp \supset Y$$

in general this inclusion is strict

Example :

$$X = L^2((0,1))$$

$$Y = C^0([0,1])$$

$$\underset{\text{by density}}{Y^\perp = \{0\}} \Longrightarrow Y^{\perp\perp} = X$$

Lemma : $(X, (\cdot,\cdot))$ Hilbert. If $Y \subset X$ subspace, then

$\boxed{Y^{\perp\perp} = \overline{Y}}$. In particular, Y closed $\Longleftrightarrow Y = Y^{\perp\perp}$.

Proof : 2^{nd} part follows straight from 1^{st} part.

For the 1^{st} part : $\quad Y^{\perp\perp} \supset Y$ (trivial by def)

$\Downarrow$ (any orthogonal is closed)

$$Y^{\perp\perp} \supset \overline{Y}$$

Assume, by contradiction that $\supset$ is strict

So: $\exists\ x_0 \in Y^{\perp\perp} \setminus \overline{Y}$

$x_0 \in Y^{\perp\perp} \subset X$

$\overline{Y}$

happening in $Y^{\perp\perp}$

$z_0 = x_0 - y_0$

$\in \overline{Y}^{\perp} = Y^{\perp} = W$

$\in Y^{\perp\perp} = W^{\perp}$ $\quad \xrightarrow[W \cap W^{\perp} = \{0\}]{} \quad \boxed{z_0 = 0}$

$\Longrightarrow \boxed{x_0 = y_0}$ contradiction. ■

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Thm. $(H, (\ ,\ ))$ Hilbert space, let

H_N subspace, $\boxed{\dim_{\mathbb{C}} H_N = N}$ $\longleftarrow$ H_N closed

and let $\{e_1, e_2, \dots, e_N\}$ be an orthonormal basis for H_N.

Claim: $x \in H$

$$\pi_{H_N}(x) = \sum_{k=1}^{N} (x, e_k)\, e_k$$

why? $\quad (x - \pi_{H_N}(x)) \perp e_k \quad \forall k = 1, \dots, N$

$(\Longleftrightarrow \perp H_N)$.