Functional Analysis 1 - L9 15/10/2020

Hilbertian bases

Def $(H, (\,,\,))$ Hilbert space. For a set I, we say that $(e_i)_{i \in I}$ is an orthonormal family if

$$(e_i, e_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

If $I = \mathbb{N}$ or $I = \mathbb{Z}$ we'll say this is an orthonormal system.

Thm. $(H, (\,,\,))$ Hilbert space, $(e_k)_{k \in \mathbb{N}}$ an orthonormal system.

a) $\forall x \in H$

$$\underbrace{\sum_{k=0}^{\infty} |(x, e_k)|^2 \leq \|x\|^2}_{\text{Bessel's inequality}}$$

b) $\forall x \in H$ the series

$$\underbrace{\sum_{k=0}^{\infty} (x, e_k) e_k}_{\text{Fourier series (of } x)}$$ converges

c) given $x \in H$ the equality $\boxed{\sum_{k=0}^{\infty} |(x, e_k)|^2 = \|x\|^2}$ happens if and only if $\sum_{k=0}^{\infty} (x, e_k) e_k = x$.

Parseval identity

Def: Setting above. We'll say that the orthonormal system (e_k) is complete (it is an Hilbertian basis) if $x = \sum_{k=0}^{\infty} (x, e_k) e_k$ $\forall x \in H$.

Proof:

a) $D_N := \operatorname{span}_{\mathbb{R}} \{e_0, e_1, \dots, e_N\}$

by last time $\|x\|^2 = \underbrace{\|\pi_{D_N}(x)\|^2}_{} + \|\pi_{D_N^\perp}(x)\|^2$

$$= \sum_{k=0}^{N} |(e_k, x)|^2 + (\;)_{\geq 0}$$

$$\Longrightarrow \quad \forall N \in \mathbb{N} \qquad \underbrace{\|x\|^2}_{\text{does not depend on } N} \geq \sum_{k=0}^{N} |(x, e_k)|^2$$

let $N \to \infty$ $\Rightarrow$ $\|x\|^2 \geq \underbrace{\sum_{k=0}^{\infty} |(x, e_k)|^2}_{\text{is summable!}}$

b) since H is complete, it's enough to check that the partial sums are Cauchy: $S_n := \sum_{k=0}^{n} (x, e_k) e_k$

take $\ell_2 \geq \ell_1$

$$\| S_{\ell_2} - S_{\ell_1} \|^2 = \left\| \sum_{k=\ell_1+1}^{\ell_2} (x, e_k) e_k \right\|^2 = \underbrace{\sum_{k=\ell_1+1}^{\ell_2} |(x, e_k)|^2}_{\text{is small if } \ell_1 \gg 1}$$

because bounded from above by the tail of a summable series

c) note that $\forall N \in \mathbb{N}$

$$\left\| x - \sum_{k=0}^{N} (x, e_k) e_k \right\|^2 \overset{\text{expanding the square}}{=} \underbrace{\|x\|^2 - \sum_{k=0}^{N} |(x, e_k)|^2}_{\geq 0 \text{ by part a)}}$$

Conclusion comes by letting $N \to \infty$ in the identity above. ■

Example: ℓ^2 "square-summable sequences"

Claim: ℓ^2 has an Hilbertian basis, given by the monomial sequences $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ (1 in the i-th slot)

orthonormal ✓

complete (?) by part c) it's enough to check that $\forall x \in \ell^2$ Parseval holds, i.e.

$$\|x\|^2 \overset{?}{=} \sum_{k=0}^{\infty} \underbrace{(x, e_k)^2}_{x_k}$$

x_k: k-th element of $x = (x_0, x_1, x_2, \dots, x_k, \dots)$

$$\|x\|^2 \underset{\text{by def. of } \ell^2 \text{ norm}}{=} \sum_{k=0}^{\infty} x_k^2$$

Prop. A Hilbert space $(H, (\,,\,))$ admits an Hilbertian basis if and only if it is $\underbrace{\text{separable}}_{\text{countable dense subset}}$.

Proof:

$\boxed{\Rightarrow}$ take (e_k) Hilbertian basis.

Claim: the set D consisting of finite linear combinations w/ coefficients in $\mathbb{Q}$ of basis elements is dense.

countable

Use $\underbrace{\text{Parseval}}_{\text{given } x \in H}$: $\forall \epsilon > 0 \ \exists N = N(\epsilon)$ such that

$$d_H^2\Big(x, \underbrace{\sum_{k=0}^{N} (x, e_k) e_k}_{\text{finite sum}}\Big) = \sum_{k=N+1}^{\infty} (x, e_k)^2 < \epsilon/2$$

approximate (x, e_k) by $q_k \in \mathbb{Q}$ s.t. $d_H^2\big(x, \sum_{k=0}^{N} q_k e_k\big) < \epsilon$.

$\boxed{\Leftarrow}$ let $(x_k)_{k \in \mathbb{N}}$ be an enumeration of a countable dense subset of H.

Make two operations:

1. $v_0 = x_0$, then given $\{v_0, v_1, \dots, v_n\}$ set $v_{n+1} = x_{k(n+1)}$

$$k(n+1) = \min\{p \in \mathbb{N} : p > k(n),\ x_p \notin \operatorname{span}_{\mathbb{R}}\{v_0, v_1, \dots, v_n\}\}$$

$$D_N := \operatorname{span}_{\mathbb{R}}\{v_0, v_1, \dots, v_N\}$$

$$D_\infty := \bigcup_N D_N \longleftarrow \text{still } \underline{\text{dense}}$$

$$\Big(\supset \bigcup_{k \in \mathbb{N}} \{x_k\}\Big)$$

2. Apply Gram–Schmidt procedure:

$$(v_k) \rightsquigarrow (e_k)$$

orthonormal system

Given $x \in H$ and $N \in \mathbb{N}$, let

$$d_N = d\,(x, D_N) = \inf_{y \in D_N} \|x - y\|$$
$$\text{where } D_N = \operatorname{span}_{\mathbb{R}}\{e_0, e_1, \dots, e_n\}$$

by density of $D_\infty = \bigcup D_N$ we have that $\forall \epsilon > 0$ $\exists N = N(\epsilon)$ s.t. $d_N < \epsilon$ but this means (by the form of projections on finite-dim subspaces)

$$\left\| x - \sum_{k=0}^{N} (x, e_k)\, e_k \right\| < \epsilon$$

Since ϵ is arbitrary, we conclude that $x = \sum_{k=0}^{\infty} (x, e_k) e_k$ i.e. (e_k) is an Hilbertian basis. ∎

<u>Cor.</u> The spaces $L^2((-\pi, \pi), \mathbb{R})$
$L^2((-\pi, \pi), \mathbb{C})$ do admit an Hilbertian basis.

<u>Thm.</u> (a) An Hilbertian basis for $L^2((-\pi, \pi), \mathbb{R})$ is given by the standard (real) trigonometric system, i.e.

$$\frac{1}{\sqrt{2\pi}}, \quad \frac{1}{\sqrt{\pi}} \cos(kx) \quad k \in \mathbb{N}_*, \quad \frac{1}{\sqrt{\pi}} \sin(kx) \quad k \in \mathbb{N}_*.$$

(b) An Hilbertian basis for $L^2((-\pi, \pi), \mathbb{C})$ is given by the standard (complex) trigonometric system, i.e.

$$\frac{1}{\sqrt{2\pi}} e^{ikx} \qquad k \in \mathbb{Z}.$$

<u>Consequences</u>:

(a) [real case]
$$S(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos(kx) + b_k \sin(kx)\right)$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(y) \cos(ky)\, dy \qquad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(y) \sin(ky)\, dy$$

Parseval's identity:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx = \frac{a_0^2}{2} + \sum_{k=1}^{\infty}(a_k^2 + b_k^2)$$

e.g. $f(x) = x \rightsquigarrow \sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}$

(Problems 5.3 – 5.4)

(b) [Complex case] $\quad S_{\mathbb{C}}(x) = \sum_{k\in\mathbb{Z}} c_k e^{ikx}$

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\, e^{-iky}\,dy$$

Parseval's identity:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx = \sum_{k\in\mathbb{Z}} |c_k|^2$$

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Rmk: an example of a non-separable Hilbert space.

$X = \{ f : [0,1] \to \mathbb{R}$ such that $f(x) = 0$ except for countably many x, and $\sum_{x\in[0,1]} |f(x)|^2 < \infty \}$

place a scalar product

$$(f,g) = \sum f(x)\, g(x)$$

complete ✓ $\Rightarrow (X, (\cdot,\cdot))$ Hilbert

but not separable (argument as for ℓ^∞).

Isomorphic Classification:

Def: Given Hilbert spaces $(H_1, (\cdot,\cdot)_1)$ and $(H_2, (\cdot,\cdot)_2)$ we'll say that a linear map $\Phi : H_1 \to H_2$ is an isometric isomorphism if it is a bijection, and $\forall\, x_1, z_1 \in H_1$

$$(x_1, z_1)_{H_1} = (\Phi(x_1), \Phi(z_1))_{H_2}.$$

Moral : classification means "classification up to isometric isomorphisms".

Def Let $(H, (\,,\,))$ be a Hilbert space. We say that an o.n. family $(e_i)_{i\in I}$ is a generalized Hilbertian basis for H if

$$\overline{\left\{ \sum_{\substack{J\subset I \\ \text{finite}}} x_i e_i \,,\; x_i \in \mathbb{K} \right\}}^{H} = H \qquad (*)$$

1 **Thm.** : Any Hilbert space admits a generalized Hilbertian basis.

2 **Thm.** $(H, (\,,\,))$, let $\mathcal{B}, \mathcal{B}'$ be two generalized Hilbertian bases, there is a bijection $\Omega : \mathcal{B} \longrightarrow \mathcal{B}'$.

$\longrightarrow$ the cardinality of any generalized H. basis is called the Hilbertian dimension of H.

3 **Thm.** Two Hilbert spaces $(H_1, (\,,\,)_1)$ and $(H_2, (\cdot,\cdot)_2)$ are isometrically isomorphic if and only if there exist generalized Hilbert bases $\mathcal{B}_1$ of $(H_1, (\cdot,\,))$, $\mathcal{B}_2$ of $(H_2, (\cdot,\,))$ having the same cardinality.

Cor. Any two separable Hilbert spaces are isometrically isomorphic, and each of them is isometrically isomorphic to ℓ^2.