Functional Analysis 1 - L10 19/10/2020

Basic Principles of Functional Analysis

Baire's Lemma $\Longrightarrow$ $\begin{cases} \cdot \text{ Uniform Boundedness Principle} \\ \cdot \text{ Open Mapping Theorem} \\ \cdot \text{ Closed Graph Theorem} \end{cases}$

1 **Theorem** (Banach-Steinhaus, or UBP) [proven in L3]

X Banach space, Y normed spaces, $(A_\lambda)_{\lambda \in \Lambda}$ family in $L(X,Y)$

w/ $\sup_{\lambda \in \Lambda} \|A_\lambda x\|_Y < \infty \quad (\forall x \in X)$

then $\sup_{\lambda \in \Lambda} \|A_\lambda\|_{L(X,Y)} < \infty$.

Application: converging family of operators.

X Banach space, Y normed space, $(A_k)_{k \in \mathbb{N}}$ w/ $A_k \in L(X,Y)$

pointwise converging i.e.

$(\forall x \in X) \quad \exists \lim_{k \to \infty} A_k x = Ax$

Then $A \in L(X,Y)$ i.e. A is a continuous (linear) operator

and $\|A\|_{L(X,Y)} \leq \liminf_{k \to \infty} \|A_k\|_{L(X,Y)} < \infty$.

Proof (UBP $\Rightarrow$ $\boxed{///}$)

(A_k) is pointwise converging $\Longrightarrow$ pointwise bounded

$\Downarrow$ UBP

$\sup_{k \in \mathbb{N}} \|A_k\|_{L(X,Y)} < \infty$

$\Downarrow$

$\exists \liminf_{k \to \infty} \|A_k\|_{L(X,Y)} < \infty$

Pick a subsequence $k \longmapsto \ell(k)$ w/

$$\|A_{\ell(k)}\|_{L(X,Y)} \xrightarrow[k\to\infty]{} \liminf_{k\to\infty} \|A_k\|_{L(X,Y)}$$

Set for $x \in X$ $Ax := \lim_{k\to\infty} A_k x$, so A is linear by basic properties of the limits. Also for $x \in X$ we can write

$$A_k x \longrightarrow Ax \underset{\text{cont of } \|\cdot\|}{\Longrightarrow} \underbrace{\|Ax\|_Y} = \lim_{k\to\infty} \|A_k x\|_Y$$

$$= \lim_{k\to\infty} \underbrace{\|A_{\ell(k)} x\|_Y}_{\leq \|A_{\ell(k)}\|_{L(X,Y)} \|x\|_X}$$

$$\leq \lim_{k\to\infty} \|A_{\ell(k)}\|_{L(X,Y)} \cdot \underbrace{\|x\|_X}$$

$$= \underbrace{\|x\|_X} \cdot \underbrace{\lim_{k\to\infty} \|A_{\ell(k)}\|_{L(X,Y)}}_{\left\{ = \liminf_{k\to\infty} \|A_k\|_{L(X,Y)} \right\}}$$

$$\Longrightarrow \|A\|_{L(X,Y)} \leq \liminf_{k\to\infty} \|A_k\|_{L(X,Y)} \quad \blacksquare$$

Rmk. (Critical role of completeness)

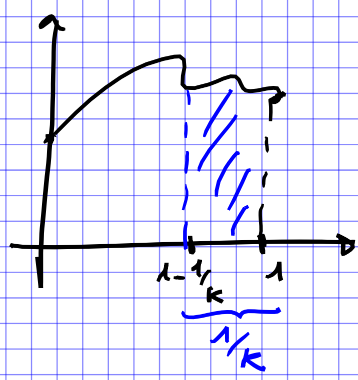
- $X = C^0([0,1])$

$\|\cdot\|_X = \|\cdot\|_{L^1}$ $\quad (X, \|\cdot\|_X)$ <u>not</u> complete

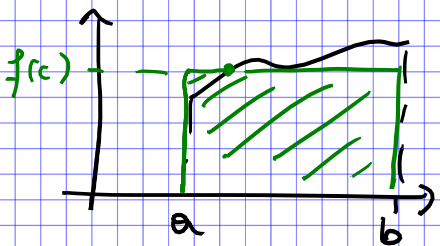
$$\overline{C^0([0,1])}^{L^1} = L^1([0,1])$$

- For $k \geq 1$ set $A_k : X \longrightarrow \mathbb{R}$

$$A_k f = k \cdot \int_{1-1/k}^{1} f(t)\,dt$$



$$\boxed{A_k f \longrightarrow f(1)} \quad (\forall f \in X)$$



$$\frac{1}{b-a}\int_a^b f(s)\,ds \underset{\uparrow}{=} f(c) \qquad c \in [a,b]$$

$$\begin{cases} A_k f = f(c_k) \\ c_k \in [1-1/k, 1] \end{cases}$$

$k \to \infty$
$c_n \to 1$

$f(c_n) \to f(1)$

$$A_n f = f(c_n) \xrightarrow{n\to\infty} f(1).$$

- $A f = f(1)$ (pointwise limit operator)

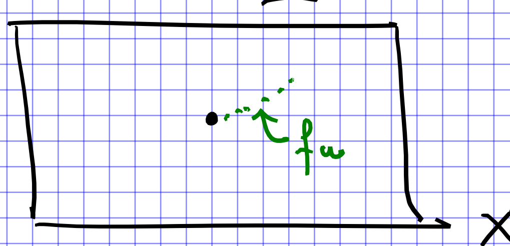
$A \notin L(X,Y)$ why? $f_n(t) = t^n$

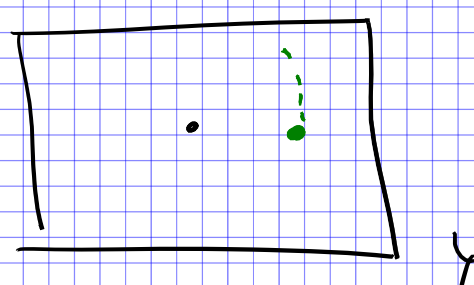
$$\boxed{A f_n = 1}$$

$$\|f_n\|_X = \|f_n\|_{L^1} = \frac{1}{n+1}$$

$$= \int_0^1 |f_n(t)|\,dt = \int_0^1 t^n\,dt = \frac{1}{n+1}$$

Conclusion: A is not continuous





(2) Open Mapping Theorem:

X, Y normed spaces, $A : X \to Y$ linear

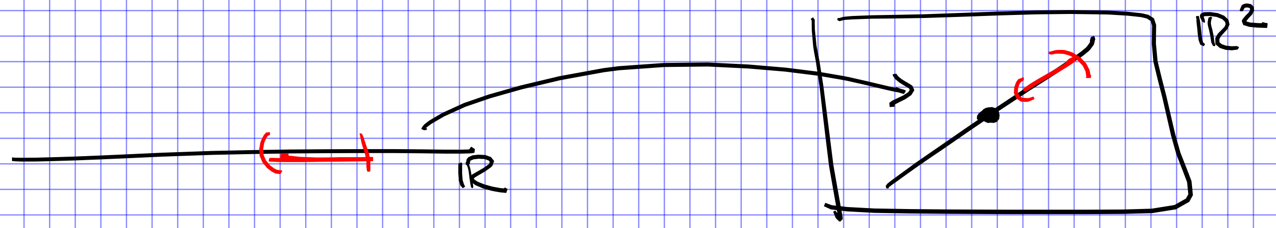
recall: A is open if $\forall\, U \subset X$ open

$A(U) \subset Y$ open.

Rmk. there are trilions of continuous maps which are not open.

$$\varphi : \mathbb{R} \to \mathbb{R}^k \qquad (k \geq 2)$$

$$\varphi(t) = (\overbrace{t, t, \dots, t}^{k\text{ times}}) \qquad C^0, \text{ not open}$$



moral: if a map is not surjective, we cannot typically expect it to be open.

Theorem (Open Mapping) Let X, Y Banach spaces
$A \in L(X, Y)$

a) if A is surjective, then A is open.

b) if A is bijective, then $A^{-1} \in L(Y, X)$
(i.e. A is a linear homeomorphism)

inverse mapping theorem

Rmk.

$$GL(X) = \{ A \in L(X), \quad A : X \circlearrowleft \text{ bijection} \}$$

Rmk. a) $\Rightarrow$ b) is trivial! Given $U \subset X$ open

$$X \underset{A^{-1}}{\overset{A}{\rightleftarrows}} Y$$

A^{-1} continuous means $(A^{-1})^{-1} U$ open in Y

A bijective $\Rightarrow$ $(A^{-1})^{-1} U = \underbrace{A U}_{\text{open in } Y \text{ by part a)}}$

$\Rightarrow A^{-1} \in L(Y, X)$.

Proof

Step 1: $\exists\, r>0 : \quad B_{2r}(0;Y) \subset \overline{A(B_1(0;X))}$

Claim

A surjective $\Longrightarrow$ $Y = \bigcup_{k\in\mathbb{N}} A(B_k(0;X))$

$\Longrightarrow$ $Y = \bigcup_{k\in\mathbb{N}} \overline{A(B_k(0;X))}$

Y complete so by Baire $\exists\, k_0$ w/

$$\mathring{\overline{A(B_{k_0}(0;X))}} \neq \emptyset$$

so $\exists\, y_0 = Ax_0$, $\exists\, r_0>0$ s.t.

$\in Y$ (A surjective)

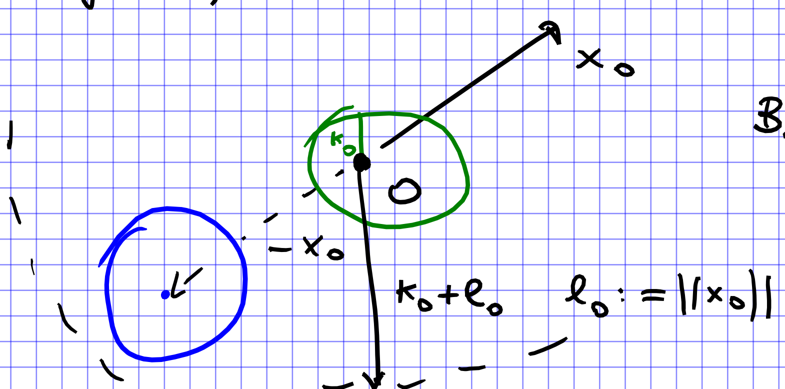
$$B_{r_0}(y_0;Y) \subset \overline{A(B_{k_0}(0;X))}$$

$$\Longleftrightarrow \quad \underbrace{y_0}_{Ax_0} + B_{r_0}(0;Y) \subset \overline{A(B_{k_0}(0;X))}$$

$$\Longleftrightarrow \quad B_{r_0}(0;Y) \subset \underbrace{\overline{A(B_{k_0}(0;X))} - Ax_0}$$

$-Ax_0$ (translation)

$$= \overline{A(B_{k_0}(0;X) - x_0)}$$



$$B_{k_0}(0;X) - x_0 \subset B_{k_0+\ell_0}(0;X)$$

Thus:

$$B_{r_0}(0;Y) \subset \overline{A(B_{k_0+\ell_0}(0;X))}$$

now scale by a factor $k_0+\ell_0$ gives the claim $\boxed{r := \frac{r_0}{2(k_0+\ell_0)}}$

$$\frac{B_{r_0}}{k_0+\ell_0}(0;Y) \subset \overline{A(B_1(0;X))}$$

Step 2 : get rid of the closure, i.e.

$$B_r(0;Y) \subset A(B_1(0;X))$$

Fix $\boxed{y \in B_r(0;Y)}$, we want to prove $y \in A(B_1(0;X))$ by constructing (x_k) s.t.

$$\begin{cases} \sum \|x_k\| < 1 \\ \underbrace{A\left(\sum_{k=0}^{n} x_k\right)}_{} \xrightarrow{n\to\infty} y \end{cases}$$

(normal conv. criterion)

A continuous $\downarrow$

Ax

$\sum x_k = x \in X$

$\|x\| < 1$

- Iterative scheme (which builds upon Step 1)

by Step 1, have $\boxed{B_r(0;Y) \subset \overline{A(B_{1/2}(0;X))}}$

so $\exists\, x_1 \in B_{1/2}(0;X)$ w/

$$\|y - Ax_1\|_Y < r/2$$

set $y_1 = \underbrace{y}_{y_0} - Ax_1$

then, given $x_1, \dots, x_k \in X$ (1)

$y_1, \dots, y_k \in Y$

w/ $\|x_\ell\| < 2^{-\ell}$ $\forall\ 1 \le \ell \le k$

$$y_\ell = y_{\ell-1} - Ax_\ell \in B_{2^{-\ell} r}$$

construct x_{k+1} by invoking Step 1 in the form

$$\underbrace{B_{2^{-k} r}(0;Y)}_{y_k} \subset \overline{A\big(\underbrace{B_{2^{-(k+1)}}(0;X)}_{x_k}\big)}$$

$\|y_k - Ax_{k+1}\| < r/2^{k+1}$, set $y_{k+1} = y_k - Ax_{k+1}$

By construction $\sum_{k=1}^{\infty} \|x_k\| < \sum_{k=1}^{\infty} 2^{-k} = 1$

$$\boxed{\left\| y - \sum_{k=1}^{n} A x_k \right\|_Y < \frac{r}{2^n}}$$

because $y_n = y - \sum_{k=1}^{n} A x_k$

$$A\left(\sum_{k=1}^{n} x_k\right) \equiv A S_n \xrightarrow[n\to\infty]{} y$$

Step 3: Fix a test set $U \subset X$, consider $AU \subset Y$
pick $y_0 \in AU$ want to get a ball round y_0 in AU.

$x_0 \in U$ w/ $A x_0 = y_0$

$s > 0$ w/ $\boxed{B_s(x_0; X) \subset U}$

Claim: $\boxed{B_{rs}(y_0; Y) \subset A(U)}$

Justify the claim: $B_{rs}(y_0; Y) = y_0 + \underbrace{B_{rs}(0; Y)}_{\text{Step 2}}$

$$\subseteq \underbrace{A x_0 + A(B_s(0; X))}_{A(B_s(x_0; X))} \subset AU$$

this is what I wanted!

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Aurelio's Dilemma: X Banach space, $Y \subset X$ closed subspace

$\pi : X \longrightarrow X/Y$, $\pi \in L(X, X/Y)$

surjective

$\Longrightarrow$ open

?: is π a 'quotient map' in the sense of General Topology?

Def: Given $\hat{X}$, $\tilde{Y}$ top. spaces, a surjective map $f:\hat{X}\to\tilde{Y}$ is called a quotient map if

$$U\subset\tilde{Y} \text{ open } \overset{(*)}{\Longleftrightarrow} f^{-1}(U)\subset\hat{X} \text{ open.}$$

Answer: YES. In our case $\pi: X\to X/_Y$ surjective,

check $(*)$.

$\boxed{\Rightarrow}$ by continuity (!)

$\boxed{\Leftarrow}$ note that (for set-theoretic reasons, since π surjective)

if $U\subset X/_Y$, $\underbrace{\pi\big(\underbrace{\pi^{-1}(U)}_{\text{open by hp.}}\big)}_{\text{open by open mapping thm.}} = U \to \underline{\text{open}}$! ∎