Functional Analysis I – L11 22/10/2020

Review / comments on open mapping theor.

$(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ } Banach $A \in L(X,Y)$

- surjective $\Rightarrow$ open
- $\underbrace{\text{bijective}}$ $\Rightarrow$ (linear) homeomorphism

(*) An important special case:

$X = Y$ as sets, may have different norms $\|\cdot\|_X$, $\|\cdot\|_Y$

$A = id : X \longrightarrow Y$ (always) linear bijection

Suppose $A \in L(X,Y)$ i.e. it is a continuous map

which means $\|x\|_Y \leq C\|x\|_X$

then (by open mapping theorem) actually $\|\cdot\|_X \overset{\sim}{=} \|\cdot\|_Y$

by which we mean $\exists\, C' > 0 : \underbrace{C'\|x\|_X \leq \|x\|_Y}_{\text{OMT}} \leq (C')^{-1}\|x\|_X$.

Examples in the class above (see below) show that completeness assumption is necessary, both for domain and the target.

Example 1: $X = Y = C^0([0,1])$

$\|\cdot\|_X = \|\cdot\|_{C^0}$, $\|\cdot\|_Y = \|\cdot\|_{L^1}$ } w/ this choice $id \in L(X,Y)$ since $\|f\|_{L^1} \leq C\|f\|_{L^\infty}$

but on the other hand it is not true that these norms are equivalent, e.g. take $f_n(t) = t^n$

$$\|f_n\|_{L^1} = \frac{1}{n+1}$$
$$\|f_n\|_{C^0} = 1$$

Moral: the OMT cannot be applied here, because $(Y, \|\cdot\|_Y)$ is not complete.

Example 2: $X = Y = \ell^2$

set $\|\cdot\|_Y = \|\cdot\|_{\ell^2}$

$\|\cdot\|_X \leftsquigarrow$ strange norm, defined using algebraic bases

separable Hilbert space, as such it has an Hilbertian basis

$\{e_i\}$ $\qquad e_i = (0, \ldots, 0, 1, 0, \ldots \ldots 0)$

$\uparrow$ i-th position

Abstract principle (Zorn's Lemma): we can extend $\{e_i\}$ to an algebraic basis $\{\beta_\iota\}_{\iota \in I}$, w/ $\|\beta_\iota\| = 1 \;\; \forall \iota \in I$

$$\text{if } \; x = \underbrace{\sum_{\iota \in I} \alpha_\iota \beta_\iota}_{\text{finite}} \qquad \|x\|_X = \sum_{\iota \in I} |\alpha_\iota|$$

$id \in \mathcal{L}(X, Y)$ since

$$\|x\|_Y := \|x\|_{\ell^2} \underset{\text{triangle}}{\leq} \sum |\alpha_\iota| \, \|\beta_\iota\| \equiv \sum |\alpha_\iota| = \|x\|_X$$

however the two norms are <u>not</u> equivalent:

$$x_k = \sum_{i=1}^{k} \frac{1}{\sqrt{k}} e_i \quad \rightsquigarrow \quad \begin{aligned} \|x_k\|_Y &= \|x_k\|_{\ell^2} = 1 \\ \|x_k\|_X &= \ldots = \sqrt{k} \end{aligned}$$

Moral: OMT is not applicable because $(X, \|\cdot\|_X)$ is <u>not</u> complete.

Closed Graph Theorem

$\left.\begin{array}{l}(X, \|\cdot\|_X)\\ (Y, \|\cdot\|_Y)\end{array}\right\}$ normed $\qquad A : X \longrightarrow Y$ linear

$$\underbrace{\Gamma_A}_{\text{graph of } A} := \{ (x, Ax) : x \in X \}$$

norm, induced by restriction by

$$\|(x,y)\|_{X\times Y} = \|x\|_X + \|y\|_Y$$

rmk. if $A \in L(X,Y)$ then $\Gamma_A \subset X \times Y$ closed.

why? $\underbrace{(x_k, Ax_k)}_{\in \Gamma_A} \longrightarrow \underbrace{(x,y)}_{? \in \Gamma_A}$

$$\left.\begin{array}{ll} x_k \longrightarrow x \implies & Ax_k \longrightarrow Ax \\ \text{but also} & Ax_k \longrightarrow y \end{array}\right\} \xRightarrow{\text{uniqueness of the limit}} Ax = y$$

$$\implies (x,y) \in \Gamma_A .$$

Thm (closed graph theorem)

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces. TFAE:

(1) $A \in L(X,Y)$

(2) $\Gamma_A \subset X \times Y$ closed.

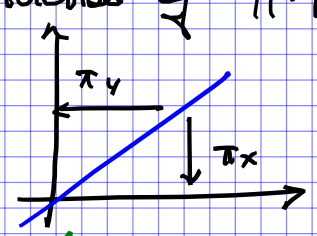
rmk. above

Pf. (2) $\implies$ (1) $\qquad (X \times Y, \|(\cdot)\|_{X\times Y})$ Banach

$\Gamma_A \subset X \times Y$ closed $\implies \Gamma_A$ complete (i.e. Banach) w/ restriction of $\|\cdot\|_{X\times Y}$

is a bijection

$$\left.\begin{array}{l} \pi_X : \Gamma_A \longrightarrow X \\ \pi_Y : \Gamma_A \longrightarrow Y \end{array}\right\} \text{both (linear and) continuous}$$



$\boxed{\text{OMT}}$ $\exists\ \pi_X^{-1} \in L(X, \Gamma_A)$

$$A = \pi_Y \circ \pi_X^{-1} \in L(X,Y) \quad \blacksquare$$

Moral: cheaper continuity check for linear operators.

$$A \in L(X,Y) \qquad\qquad \Gamma_A \text{ closed}$$

means:

$$x_k \longrightarrow x \implies \begin{cases} Ax_k \longrightarrow y \\ Ax = y \end{cases} \qquad \left.\begin{matrix} x_k \longrightarrow x \\ Ax_k \to y \end{matrix}\right\} \implies y = Ax$$

Application (Töplitz criterion): let $(H, \langle\cdot,\cdot\rangle)$ be Hilbert space, and $A: H \circlearrowleft$ linear and symmetric i.e.

$$\langle Ax, y\rangle = \langle x, Ay\rangle \quad \forall x, y$$

then $A \in L(X,Y)$. "symmetric operators are automatically continuous".

Proof (Töplitz)

Like in the right column above $\quad x_k \longrightarrow x$, $\quad Ax_k \longrightarrow y$

$$\langle Ax_k, z\rangle = \langle x_k, Az\rangle \quad (\forall k)$$

$$\downarrow \qquad\qquad \downarrow$$

$$\langle y, z\rangle = \langle x, Az\rangle$$

$$= \langle Ax, z\rangle \implies \langle y - Ax, z\rangle = 0 \quad (\forall z \in H)$$

$$\implies \|y - Ax\|^2 = 0$$

$$(z = y - Ax) \qquad \implies y = Ax. \qquad \blacksquare$$

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Introduction to unbounded operators

$\left.\begin{matrix}(X, \|\cdot\|_X) \\ (Y, \|\cdot\|_Y)\end{matrix}\right\}$ normed

in many natural situations

$$A: \underbrace{D(A)}_{\text{vector subspace}} \subseteq X \longrightarrow Y$$

and $D(A) \subsetneq X$, in which case the operator is called unbounded.

Example: $X = C^0([0,1])$

$\|\cdot\|_X$ ↗ usual C^0 norm

$$\frac{d}{dt}: D(A) \subset X \longrightarrow X$$

$$D(A) = C^1([0,1])$$

Claim: 1. $\frac{d}{dt} \notin L(D(A), X)$

2. $\Gamma_A = \{(x, Ax) : x \in D(A)\} \subset X \times X$ closed

Fact 1: $f_n(t) = t^n$ $\quad \|f_n\|_X = 1 \quad \forall n \geq 1$

$f_n'(t) = n t^{n-1}$ $\quad \|f_n'\|_X = \|A f_n\| = n \quad \forall n \geq 1$

if A were continuous (i.e. if $A \in L(D(A), X)$) then $\exists\, C > 0$

$\|A f_n\|_X \leq C \|f_n\|_X$, which is clearly false.

Fact 2: Γ_A closed means

$$(f_n, f_n') \longrightarrow (f, g)$$

then $(f, g) \in \Gamma_A$ i.e. $f \in C^1([0,1])$

$f' = g$ (cf. Analysis I/II)

Moral: the two conditions above are not equivalent if $D(A) \subsetneq X$.

Continuous Inverse thm:

$\left.\begin{array}{l}(X, \|\cdot\|_X) \\ (Y, \|\cdot\|_Y)\end{array}\right\}$ Banach spaces $\qquad A: D(A) \longrightarrow Y$

linear, bijective

w/ Γ_A closed.

then $\exists\, B: Y \longrightarrow D(A)$

$\in L(Y, D(A))$ w/

$$\begin{cases} BA = \mathrm{id}|_{D(A)} \\ AB = \mathrm{id}_Y \end{cases}$$

Proof: $\pi_Y : \Gamma_A \longrightarrow Y$

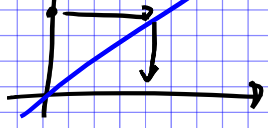
bijective, linear, continuous

($\Leftarrow$ since A is bijective)

so (closed graph theorem) $\exists\, \pi_Y^{-1} \in L(Y, \Gamma_A)$

Set $B = \pi_X \circ \pi_Y^{-1}$

$\in L(Y, D(A))$ $\blacksquare$



Application to a concrete case:

$X = C^0([0,1])$ $\qquad D(A) = C^1_0([0,1])$

$\|\cdot\|_X$ C^0 norm $\qquad = \{ f \in C^1([0,1]),\ f(0) = 0 \}$

$\frac{d}{dt} : D(A) \longrightarrow X$

- linear
- has closed graph (as done)
- injective
- surjective: preimage of $h \in X$ is $f(t) = \int_0^t h(s)\,ds$

Continuous Inverse Theorem $\Big\}$ $\exists$ an inverse $B \in L(X, D(A))$

$$Bh(t) = \int_0^t h(s)\,ds$$

functional interpretation of indefinite integral as antiderivative

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Preface to the next lecture:

Thm. $\Omega \subset \mathbb{R}^n$ open, $f \in L^1_{loc}(\Omega)$ $\qquad$ true if $f \in L^p(\Omega)$ $\forall p \in [1,\infty]$

If $\int_\Omega f\varphi = 0 \quad \forall \varphi \in C^\infty_c(\Omega) \implies f = 0$ $\mathcal{L}^n$-a.e.

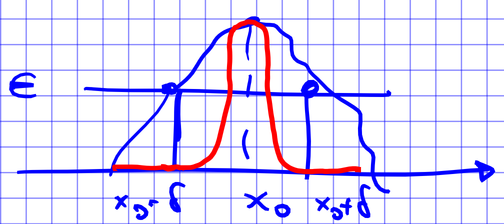
Rovel: what if $f \in C^0(\Omega)$??

by contradiction if $f \neq 0$

then wlog $f(x_0) > 0$

so by cont. $\exists \delta > 0$ w.

$x \in [x_0 - \delta, x_0 + \delta]$ $\quad f(x) \geq \epsilon$



take $\varphi \in C^\infty_c(\Omega)$ w/ support $\qquad \int f\varphi > 0 \quad$ ↯

In general (no cont. hp. on f) there are two ways to do the proof:

- by convolution
- by $\underbrace{\text{Lebesgue differentiation theorem}}$

$f \in L^1_{loc}(\Omega) \implies f$ is "continuous in mean" $\mathcal{L}^n$-a.e., that is

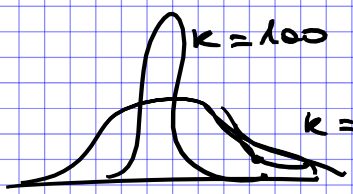
$$\lim_{r \downarrow 0} \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} |f(x) - f(x_0)|\, dx = 0$$

Given $x_0 \in \Omega$ pick a radial cutoff function $\varphi \in C_c^\infty(B_r(0))$

$$\int_{B_r(0)} \varphi\, dx = 1$$

valid test function

$$\varphi_k(x) = k^n \varphi(k(x - x_0))$$



$$0 = \int_\Omega f \varphi_k \qquad \left(\int \varphi_k = 1\ \forall k\right)$$

$$= \underbrace{\left[\int_\Omega (f(x) - f(x_0))\, \varphi_k\right]}_{\le C k^n \left(\frac{r}{k}\right)^n \underset{\uparrow\ \text{Lebesgue}}{o(1)}} + f(x_0)$$

$$\implies f(x_0) = 0 \quad \mathcal{L}^n \text{ a.e.} \qquad \blacksquare$$