Functional Analysis 1 — L12 26/10/2020

Closable operators:

$(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ } normed spaces, $\Gamma \subset X \times Y$ linear subspaces

Def: Γ is called a linear graph if it is a linear subspace which is also a graph, i.e. if

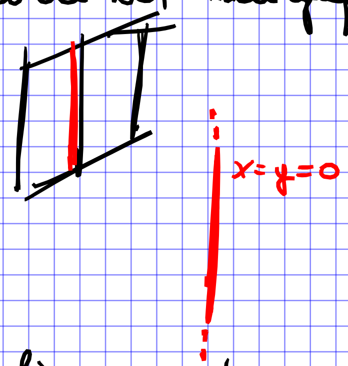
$$(*) \quad \begin{cases} (x, y_1) \\ (x, y_2) \end{cases} \in \Gamma \implies y_1 = y_2 .$$

(rmk. $(*)$ is equivalent to

$$(*)' \quad (0, y) \in \Gamma \implies y = 0 \ )$$

Examples: in $\mathbb{R}^3$, what are the (proper) linear subspaces that are NOT linear graphs?

2 classes:
- planes containing the z-axis
- the line corresponding to the z-axis



Rmk. $A : D(A) \longrightarrow Y$ linear $\implies \Gamma_A$ linear graph.

Conversely, if $\Gamma \subset X \times Y$ is a linear graph then we can associate to it a unique linear map $A : D(A) \longrightarrow Y$ by $D(A) = \pi_X(\Gamma)$ and $Ax := \pi_Y\left((\{x\} \times Y) \cap \Gamma \right)$.

Def. given $A : D(A) \longrightarrow Y$ (linear)
$B : D(B) \longrightarrow Y$ w/ $D(A) \subset D(B) \subset X$
we'll say that B is an extension of A if $B|_{D(A)} = A$.

Def.: given $A : D(A) \longrightarrow Y$ (linear) we call closure of A, denoted $\overline{A}$, the operator that is uniquely associated to $\overline{\Gamma_A} \subset X \times Y$, whenever $\overline{\Gamma_A}$ is a linear graph.

Rmk. : assuming A is closable, we always have

$$D(A) \subset D(\bar{A}) \subset \overline{D(A)}$$

(see below, cf. FA 2)

both may be strict

$$D(\bar{A}) = \{ x \in X : \exists (x_k) \in D(A) \text{ w/ } x_k \to x \text{ and } \exists y \in Y \text{ w/ } Ax_k \to y \}$$

"projections on 1st factor of limit points along Γ_A"

Q. when is a given lin. oper. A closable ??

Prop. (tautology) $A : D(A) \to Y$ is closable if and only if for $((x_k, y_k)) \subset \Gamma_A$ one has

$$\left. \begin{array}{l} x_k \to 0 \\ \text{and } \; y_k = Ax_k \to y \end{array} \right\} \Longrightarrow \boxed{y = 0}$$

Thm. if $A \in L(D(A), Y)$ then A is closable.

Pf. (use criterion above : "it's enough to check A closable at the origin")

$$A \in L(D(A), Y) \iff \underbrace{\sup_{\substack{x \in D(A) \\ x \neq 0}} \frac{\|Ax\|_Y}{\|x\|_X}}_{=: \|A\|} < \infty$$

so taken a sequence (x_k) w/ $x_k \in D(A) \; \forall k$, $x_k \to 0$

$$\|Ax_k\|_Y \leq \underbrace{\|A\| \, \|x_k\|_X}_{\to 0} \Longrightarrow \boxed{\|Ax_k\|_Y \to 0}$$

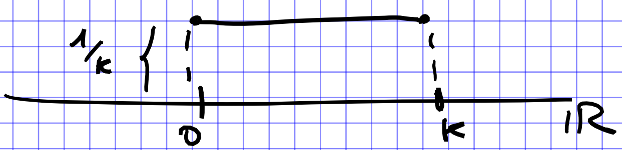
so if $y = \lim_{k \to \infty} Ax_k \Longrightarrow y = 0$ ∎

Example: a non-closable operator

$X = L^2(\mathbb{R})$ $\qquad Y = \mathbb{R}$

$D(A) = L^2_c(\mathbb{R})$ $\qquad A : f \longmapsto \int_{-\infty}^{+\infty} f(t)\,dt$

- A is linear
- A is <u>not</u> closable $\qquad f_k = \frac{1}{k} \chi_{[0,k]}$

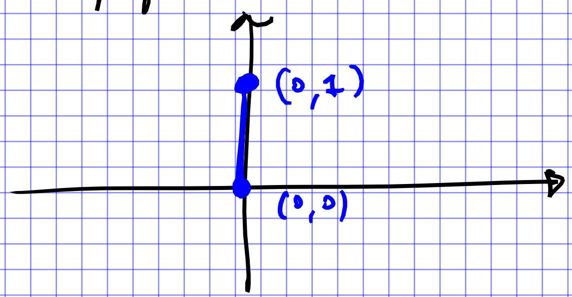


$\| f_k \|_{L^2} = 1/\sqrt{k}$ $\quad$ so $\quad \|f_k\|_{L^2} \xrightarrow{k \to \infty} 0$

but $\quad A f_k = 1 \quad \forall k$

So $\underbrace{(f_k, A f_k)}_{\text{honest points on the graph } \Gamma_A} \xrightarrow{\text{as pairs in } X \times Y} (0, 1)$

A is not closable because



Example: a closable operator

$\Omega \subset \mathbb{R}^n$ open, bounded domain

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$$

$$\Delta : \underbrace{C_c^\infty(\Omega)}_{D(A)} \subset L^2(\Omega) \longrightarrow L^2(\Omega)$$

<u>Next</u>:
- Δ is closable
- Δ is <u>not</u> bounded.

Let $u_k \in D(\Delta)$ $\quad u_k \xrightarrow[L^2]{} 0$, $\quad f_k = \Delta u_k \xrightarrow[L^2]{} f$

Pick a "test function", i.e. $\varphi \in \underbrace{C^\infty_c(\Omega)}_{\text{test functions}}$

$$\underbrace{\int_\Omega f_k \varphi}_{\displaystyle\downarrow\ \int_\Omega f\varphi} = \int_\Omega (\Delta u_k)\,\varphi \overset{\text{int by parts}}{=} \underbrace{\int_\Omega u_k\, \Delta\varphi}_{\displaystyle\downarrow\ 0}$$

$$\left(\left|\int_\Omega u_k\, \Delta\varphi\right| \le \underbrace{\|u_k\|_{L^2}}_{\to 0\ (k\to\infty)} \|\Delta\varphi\|_{L^2} \right)$$

(why?)

$$\underbrace{\int_\Omega f_k\varphi}_{\displaystyle\downarrow\ \int_\Omega f\varphi} = \underbrace{\int_\Omega (f_k - f)\,\varphi}_{\displaystyle\downarrow\ 0 \text{ true by Hölder}} + \int_\Omega f\varphi$$

$$\left(\left|\int_\Omega (f_k - f)\,\varphi\right| \le \underbrace{\|f_k - f\|_{L^2}}_{\to 0} \|\varphi\|_{L^2} \right)$$

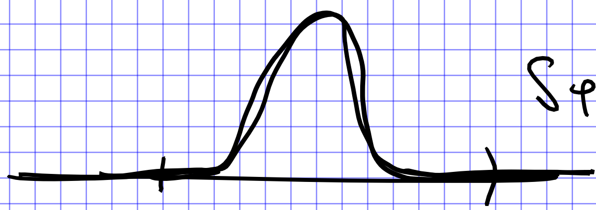
Conclusion: $\displaystyle\int_\Omega f\varphi = 0 \quad \forall \varphi \in C^\infty_c(\Omega)$

$$\xRightarrow{\ \ }_{\text{(last time)}} \quad f = 0 \quad \mathcal{L}^n\text{-a.e.}$$

(in part $f = 0$ in $L^2(\Omega)$)

$\Longrightarrow$ $\underbrace{\Delta : C^\infty_c(\Omega) \subset L^2(\Omega) \longrightarrow L^2(\Omega)}$ is closable.

Check: this op. is not bounded, so the example above is non-trivial.



$\int \varphi^2(y)\,dy = 1$

Fix $\varphi \in C^\infty_c(B_1)$

$$\varphi_\epsilon(x) = \epsilon^{-\alpha}\, \varphi(x/\epsilon)$$

we choose $\alpha > 0$ so to have "conservation of L^2 norm" i.e.

$$\int |\varphi_\epsilon|^2 = 1 \quad (\forall \epsilon > 0)$$

$$\underbrace{\int |\varphi_\epsilon|^2}_{= 1} = \epsilon^{-2\alpha} \int \varphi^2(x/\epsilon)\, dx$$

$(\forall \epsilon > 0)$

$$\underset{\text{" } y = x/\epsilon \text{ "}}{=} \epsilon^{-2\alpha} \int \varphi^2(y)\, \epsilon^n\, dy = \epsilon^{n-2\alpha} \int \varphi^2(y)\, dy$$

$\hookrightarrow \epsilon^{n-2\alpha} = 1$

$n = 2\alpha$ $\boxed{\alpha = n/2}$

if Δ were bounded i.e. $\Delta \in \mathcal{L}(C_c^\infty, L^2)$

then $\exists\, C > 0$ $\quad \|\Delta u\|_{L^2} \leq C \|u\|_{L^2}$

in part. $\quad \|\Delta \varphi_\epsilon\|_{L^2} \leq C \underbrace{\|\varphi_\epsilon\|_{L^2}}_{\equiv 1}$

$$\begin{array}{c} \| \\ \epsilon^{-2} \|\varphi_\epsilon\|_{L^2} \\ \| \\ \epsilon^{-2} \cdot 1 \end{array}$$

let $\epsilon \to 0$ get a contradiction!

Same story for " general " differential operators:

$$A : C_c^\infty(\Omega) \subset L^p(\Omega) \longrightarrow L^p(\Omega)$$

$$Au = \sum_{|\alpha| \leq N} a_\alpha(x) D^\alpha u$$

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n} \qquad \Omega \subset \mathbb{R}^n$$

$$\alpha = (\alpha_1, \ldots, \alpha_n) \qquad |\alpha| = \alpha_1 + \cdots + \alpha_n$$

(for Δ : $\underbrace{N = 2}_{\text{order of the diff. op.}}$ $\quad \alpha \in \{(2, 0, \ldots, 0), (0, 2, 0, \ldots, 0), \ldots (0, 0, 0, \ldots, 2)\}$)

Assume $\boxed{a_\alpha \in C^N(\overline{\Omega})}$

<u>Thm.</u> any differential operator as above is closable.

Pf. $u_k \in C_c^\infty(\Omega)$ $\quad u_k \xrightarrow{L^p} 0$

$$A u_k = f_k \xrightarrow{L^p} f$$

As above, given any $\varphi \in C_c^\infty(\Omega)$

$$\underbrace{\int_\Omega f_k \varphi}_{\downarrow \int_\Omega f\varphi} = \int_\Omega (A u_k)\varphi = \int_\Omega \sum_{|\alpha|\le N} a_\alpha(x) D^\alpha u_k \varphi$$

$$\boxed{= \underbrace{\int_\Omega \sum_{|\alpha|\le N} (-1)^{|\alpha|} u_k D^\alpha(a_\alpha \varphi)}}$$

$$|\quad| \le \|u_k\|_{L^p} \| \underbrace{D^\alpha(a_\alpha\varphi)}_{\in C^0(\bar\Omega)} \|_{L^q}$$

so $\|D^\alpha(a_\alpha\varphi)\|_{L^q} \le \bar{C}$ independent of k

$\longrightarrow 0$

As above, $\int_\Omega f\varphi = 0$

$$\forall \varphi \in C_c^\infty(\Omega)$$

Conclusion: $f = 0$ $\mathcal{L}^n$-a.e.

(in part $f = 0$ in $L^p(\Omega)$) $\Longrightarrow$ A closable.

Hahn-Banach and Duality

First part: X vector space over $\mathbb{k} = \mathbb{R}$.

<u>Def</u>. $p : X \longrightarrow \mathbb{R}$ is called <u>sublinear</u> if

i) $p(\alpha x) = \alpha\, p(x) \quad \forall x \in X, \ \forall \alpha \geq 0$.

ii) $p(x+y) \leq p(x) + p(y) \quad \forall x, y \in X$

<u>Ex.</u> If $(X, \|\cdot\|)$ normed space, then $\|\cdot\| : X \longrightarrow \mathbb{R}$ is sublinear.

<u>Theorem</u> (Hahn-Banach): Let M be a linear subspace of a vector space X, $p : X \longrightarrow \mathbb{R}$ sublinear and $f : M \longrightarrow \mathbb{R}$ <u>linear</u> and $f(x) \leq p(x)$ $\forall x \in M$

Then $\exists$ $F : X \longrightarrow \mathbb{R}$ <u>linear</u> such that

$$F|_M = f$$

$$F(x) \leq p(x) \quad \forall x \in X.$$

<u>Corollary</u> (dominierte Fortsetzung)

Let $(X, \|\cdot\|_X)$ be a normed space and let $f : M \to \mathbb{R}$ linear and continuous (here $M \subset X$ linear subspace).

Then $\exists$ $F \in \mathcal{L}(X; \mathbb{R})$ such that

$$F|_M = f$$

$$\|F\|_{\mathcal{L}(X;\mathbb{R})} = \|f\|_{\mathcal{L}(M;\mathbb{R})}.$$

<u>Pf.</u> (corollary given thm)

pick $p(x) := \|x\|_X \cdot \|f\|_{\mathcal{L}(M;\mathbb{R})}$

<u>check</u>: p is sublinear. Also $\underbrace{f(x) \leq p(x)}$ $\forall x \in X$

true by def of $\|f\|_{L(M;\mathbb{R})} = \sup\limits_{\substack{x\in M \\ x\neq 0}} \dfrac{|f(x)|}{\|x\|_X}$

So, thm. tells us $\exists$ $F : X \longrightarrow \mathbb{R}$ linear w/

$$F(x) \leq \|x\|_X \, \|f\|_{L(M;\mathbb{R})}$$

$$\Longrightarrow \quad \|F\|_{L(X;\mathbb{R})} \leq \|f\|_{L(M;\mathbb{R})}$$

but on the other hand, the converse inequality

$$\|F\|_{L(X;\mathbb{R})} \geq \|f\|_{L(M;\mathbb{R})}$$

is always true.

<u>Conclusion</u>: $\|F\|_{L(X;\mathbb{R})} = \|f\|_{L(M;\mathbb{R})}$ $\blacksquare$