Functional Analysis I – L13 29/10/2020

Hahn-Banach (real case)

Theorem X be a vector space $/\mathbb{R}$, $M \subset X$ linear subspace
$p: X \longrightarrow \mathbb{R}$ sublinear, $f: M \longrightarrow \mathbb{R}$ linear and dominated i.e. $f(x) \le p(x)$ $\forall x \in M$.

Then $\exists$ $F: X \to \mathbb{R}$ linear such that:

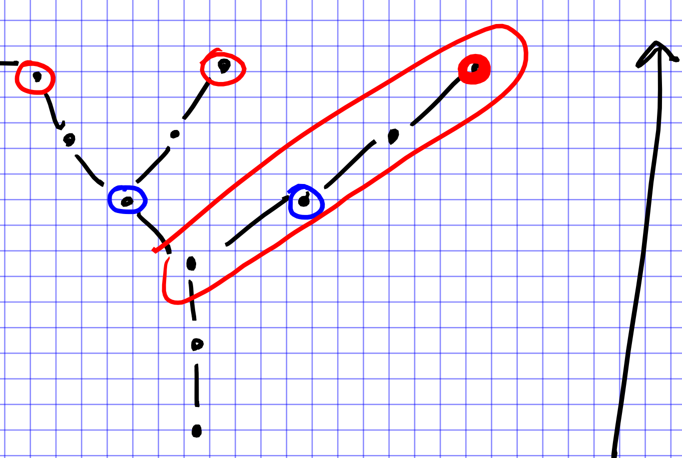
- $F|_M = f$
- $F(x) \le p(x)$ $\forall x \in X$.

Corollary : "isometric extension of linear functionals"

Pf. of Theorem: we'll argue by transfinite induction, use

Zorn's Lemma: Let $(P, \le)$ be a non-empty partially ordered set satisfying the $\underbrace{\text{chain condition}}_{\text{every totally-ordered subset has an upper bound}}$. Then $(P, \le)$ has maximal elements.

example



order: $a \le b$ iff b lies higher (or at the same height) as a

Proof : declare

$\mathcal{P} = \{ (N, h)$: $M \subseteq N \subseteq X$ linear subspace
$h: N \to \mathbb{R}$ linear, • $h|_M = f$
•• $h(x) \le p(x)$ $\forall x \in N \}$

$\leq$: $(N_1, h_1) \leq (N_2, h_2)$ if $N_1 \subseteq N_2$
and $h_2|_{N_1} = h_1$

"extension property"

· **check**: $(\mathcal{P}, \leq)$ satisfies the chain condition.

consider $\mathcal{P}'$ a totally-ordered subset of $\mathcal{P}$ (w.r.t. $\leq$).

wlog $\mathcal{P}' = \bigcup_{i \in I} \{(N_i, h_i)\}$

upper bound: $N_* := \bigcup_{i \in I} N_i$

$h_* := \bigcup_{i \in I} h_i$ (*)

(*) in the sense that $h_*(x) = h_i(x)$ if $x \in N_i$

Well-posed: if $x \in N_i \cap N_k$ then (by total order wlog.)

$N_i \subseteq N_k$ and $h_i = h_k$ on N_i, in part.

$h_i(x) = h_k(x)$.

Thus Zorn's Lemma applies and gives a maximal element

$(N, g) \in \mathcal{P}$.

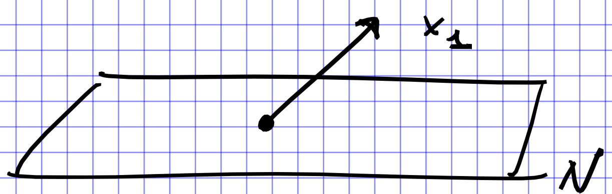
Claim: $N = X$ and so $g: X \to \mathbb{R}$

by contradiction $N \subsetneq X$ so $\exists\, x_1 \in X \setminus N$. We'll now show

that $(N \oplus \langle x_1 \rangle, g_1) \in \mathcal{P}$, violating the maximality

of (N, g).

Idea:



To close the proof, we need to define $g_1 : N \oplus \langle x_1 \rangle \longrightarrow \mathbb{R}$ extending g, so

$$g_1(x + \lambda x_1) = g_1(x) + \lambda g_1(x_1) = g(x) + \lambda \underbrace{g_1(x_1)}_{\alpha}$$

$x \in N$, $\lambda \in \mathbb{R}$

need to choose α so that
$$\boxed{\begin{array}{l} g_1(x + \lambda x_1) \leq p(x + \lambda x_1) \\ \forall x \in N, \forall \lambda \in \mathbb{R} \end{array}}$$

Morally speaking: from a necessary condition is

$$\underbrace{g_1(x_1)}_{\alpha} \geq g(x) - p(x - x_1) \quad \forall x \in N$$

temptation: define $\alpha := \sup_{x \in N} g(x) - p(x - x_1)$

Issues: $\alpha \overset{?}{\in} \mathbb{R}$, does it do the job?

Trick: $\forall x, y \in N$

$$g(x) + g(y) = g(x - x_1 + y + x_1)$$
$$\leq p(x - x_1 + y + x_1) \leq p(x - x_1) + p(y + x_1)$$

$$\underbrace{g(x) - p(x - x_1)}_{\text{does not depend on } y} \leq \underbrace{p(y + x_1) - g(y)}_{\text{does not depend on } x} \qquad (\$)$$

thus $\underbrace{\sup_{x \in N} g(x) - p(x - x_1)}_{\alpha} < \infty$

This does the job:

(1) $g_1(x - x_1) \leq p(x - x_1) \quad \forall x \in N$

(2) $g_1(y + x_1) \leq p(y + x_1) \quad \forall y \in N \qquad (\$)$

to gain $\boxed{(4)}$ for any $x \in \mathbb{R}$, we argue by scaling:

apply (1) w/ $t^{-1}x$ in lieu of x $(t>0)$

$$g_1(t^{-1}x - x_1) \leq p(t^{-1}x - x_1)$$

multiply by $t>0$ $\quad g_1(x - tx_1) \leq p(x - tx_1)$

Similarly by applying (2) w/ $t^{-1}y$ in lieu of y

$$(\ldots) \qquad g_1(y + tx_1) \leq p(y+tx_1) \quad (t>0).$$

This provides the desired extension of g to $\mathcal{N} \oplus \langle x_1 \rangle$ thus violating maximality. Contradiction $\Rightarrow$ proof is complete. ∎

Complex Version: X vector space / $\mathbb{C}$

Def: $p: X \longrightarrow \mathbb{R}$ is called $\mathbb{C}$-sublinear if

i) $p(\alpha x) = |\alpha|\, p(x) \quad \forall x \in X \quad \forall \alpha \in \mathbb{C}$

ii) $p(x+y) \leq p(x) + p(y) \quad \forall x, y \in X$

Rem R.: (*) $p: X \to \mathbb{R}$ is $\mathbb{C}$-sublinear $\Rightarrow p: X \to \mathbb{R}$ is $\mathbb{R}$-subl.

(**) $\quad 0 = p(0) = p(x-x) \leq p(x) + p(-x) = 2p(x)$

so $p: X \to \mathbb{R}$ $\mathbb{C}$-sublinear is non-negative definite.

Theorem (Hahn-Banach over $\mathbb{C}$)

Let X be a vector space over $\mathbb{C}$, $M \subset X$ be a $\mathbb{C}$-linear subspace $f: M \longrightarrow \mathbb{C}$ $\mathbb{C}$ linear, and dominated i.e.

$$\forall x \in M \quad |f(x)| \leq p(x)$$

for some $\mathbb{C}$-sublinear map p. Then there is an extension $F: X \longrightarrow \mathbb{C}$ w/

- $F|_M = f$
- $|F(x)| \leq p(x) \quad \forall x \in M$.

Corollary:

Proof of Thm. (reduce to invoke the real version)

Given f, decompose it as $f = \underbrace{f_1}_{Re(f)} + i \underbrace{f_2}_{Im(f)}$

Check: $\underbrace{Re(f) = f_1 \text{ is } \mathbb{R}\text{-linear}}$, and $\underbrace{f_1 \leq p}$

f $\mathbb{C}$-linear $\Longrightarrow$ $f(\lambda x) = \lambda f(x) \quad \forall \lambda \in \mathbb{R}$

$|f_1(x)| \leq |f(x)| \leq p$ (hyp.)

$$f_1(\lambda x) + i f_2(\lambda x) = \lambda \left[f_1(x) + i f_2(x)\right]$$
$$= \lambda f_1(x) + i \lambda f_2(x)$$

$\Longrightarrow$ (look at the real part) $f_1(\lambda x) = \lambda f_1(x)$

so by HB $\exists$ $F_1 : X \longrightarrow \mathbb{R}$ $\underbrace{\mathbb{R}\text{-linear}}$

w/
- $F_1|_M = f_1$
- $F_1(x) \leq p(x) \quad \forall x \in X$

Set $F : X \longrightarrow \mathbb{C}$

by $\boxed{F(x) := F_1(x) - i F_1(ix)} \quad \forall x \in X$

this functional does the job

a) F is $\mathbb{R}$-linear (since F_1 is)

b) $F(ix) = i F(x)$

(check: $F(ix) = F_1(ix) - i F_1(i \cdot ix)$
$$= F_1(ix) + i F_1(x)$$
$$= i\,(F_1(x) - i F_1(ix)) = i F(x))$$

c) F is $\mathbb{C}$ - linear (just combine a) and b))

d) $F|_M = f$ $\quad [f = f_1 + i f_2]$

$x \in M \quad F(x) = F_1(x) - i\, F_1(ix)$

$= f_1(x) - i\, f_1(ix)$

so it's enough to check that $\boxed{f_2(x) = -f_1(ix)}$

This follows from $\mathbb{C}$- linearity of f, specified to i

i.e. $\underbrace{f(ix) = i\, f(x)}$

$\Leftrightarrow f_1(ix) + i\, f_2(ix) = i\,(f_1(x) + i\, f_2(x))$

$= i\, f_1(x) - f_2(x)$

$= -f_2(x) + i\, f_1(x)$

$\Longrightarrow f_1(ix) = -f_2(x)$

e) $|F(x)| \leq p(x) \quad \forall x \in X$

given $x \in X$, write $F(x) = |F(x)|\, e^{-i\theta}$

$$\underbrace{|F(x)|}_{\in \mathbb{R}} = \underbrace{e^{i\theta}}_{\alpha} F(x) \underset{\mathbb{C}\text{-linear}}{=} \underbrace{F(\alpha x)}_{\in \mathbb{R}} = F_1(\alpha x)$$

$\alpha \in \mathbb{C},\ |\alpha| = 1$

$\leq p(\alpha x)$

by construction (real HB)

F_1 is dominated by p

$= |\alpha|\, p(x) = p(x)$

axiom i) of $\mathbb{C}$-sublinear

Conclusion : $|F(x)| \leq p(x) \quad \forall x \in X$ ∎