Functional Analysis I — L14 2/11/2020

Duality

- today: general facts about duality in Banach spaces
- next: " " " " in Hilbert spaces

$(X, \|\cdot\|_X)$ be a normed space.

Def: We set $X^* := L(X, \mathbb{R})$ dual space of X.

This space is always a Banach space (even when X is not), w/ norm

$$\|\ell\|_{X^*} = \sup_{\|x\| \le 1} |\ell(x)|$$

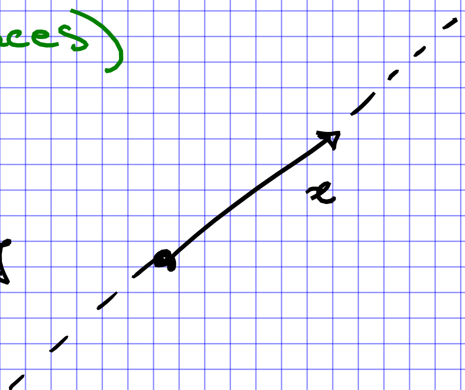
Notation: $x^* \in X^*$, we'll write $\underbrace{\langle x^*, x\rangle}$ for $x^*(x)$ — will get a meaning after next class

Prop. (abundance, $\exists$ of aligned functional(s)).

given $x \in X$ $\exists$ $x^* \in X^*$ w/ $\langle x^*, x\rangle = \|x^*\|^2_{X^*} = \|x\|^2_X$.

(terminology: suggested by Hilbert spaces)

Pf. (use Hahn-Banach)



$$M = \operatorname{span}_{\mathbb{R}}\{x\} = \{tx,\ t \in \mathbb{R}\}$$

define $f: M \to \mathbb{R}$ by

$$f(tx) = t\,\|x\|^2_X \qquad [\, f(x) = \|x\|^2_X \,]$$

$$f \in L(M; \mathbb{R}) \quad \text{w/} \quad \|f\|_{L(M,\mathbb{R})} = \sup_{\|tx\|_X \le 1} |f(tx)|$$

$$= \sup_{\|tx\|_X \le 1} |t|\,\|x\|^2_X = \underbrace{\sup_{\|tx\|_X \le 1} \|tx\|_X}_{=1} \cdot \|x\|_X = \|x\|_X .$$

Now $\exists$ isometric extension $x^* \in L(X;\mathbb{R})$

$$\begin{cases} \|x^*\|_{X^*}^2 = \|x\|_X^2 \\ \langle x^*, x\rangle = f(x) = \|x\|_X^2 \end{cases}$$

■

Dual Characterization of the norm

recall:
$$\|x^*\|_{X^*} = \sup_{\|x\|_X \leq 1} |\langle x^*, x\rangle|$$

now:
$$\|x\|_X = \sup_{\|x^*\|_{X^*} \leq 1} |\langle x^*, x\rangle| \qquad (*)$$

Pf: if $x = 0$ then nothing to prove!

else, assume WLOG $\|x\|_X = 1$

(#) note that $\|x^*\|_{X^*} \leq 1 \implies |\langle x^*, x\rangle| \leq \underbrace{\|x\|_X}_{=1}$

but by previous prop. we know that given $x \in X$

$\exists\ \check{x}^*$ w/ $|\langle \check{x}^*, x\rangle| = \|x\|_X^2 = \|\check{x}^*\|_{X^*}^2$, where $\|x\|_X^2 = \|x\|_X = 1$ and $\|\check{x}^*\|_{X^*}^2 = \|\check{x}^*\|_{X^*} = 1$

(#) it follows that $\displaystyle\sup_{\|x^*\|_{X^*} \leq 1} |\langle x^*, x\rangle| \geq 1$

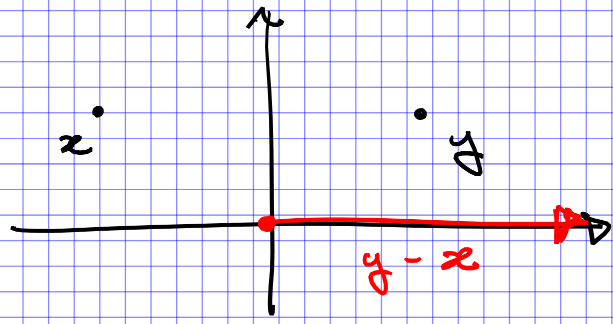
the two inequalities together imply equality. ■

~ o ~

Some (first) geometric implications of Hahn-Banach:

① Separation of Points:

Given $x \neq y$ in X $\exists\ \ell \in X^*$ w/ $\ell(x) \neq \ell(y)$



functional : 2^{nd} coordinate

$\hookrightarrow \ \ell(x) = \ell(y)$

functional : 1^{st} coordinate

$\hookrightarrow \ \ell(x) \neq \ell(y)$

Pf. if $x \neq y$ consider $y - x \neq 0$ in X

appeal to existence of an aligned functional for $y - x$

so $\exists\ \ell \in X^*$ w/ $\ell(y-x) = \|y-x\|^2 > 0$

$$\ell(y-x) = \ell(y) - \ell(x)$$

$$\Longrightarrow \ \ell(y) > \ell(x) \quad \blacksquare$$

② Separation of a point from a closed subspace:

Let $M \subset X$ be a closed (linear) subspace, $M \subsetneq M$

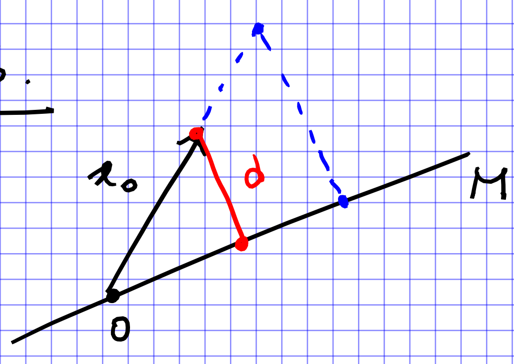
and pick $x_0 \notin M$.

Set $d = \operatorname{dist}(x_0, M) = \inf_{x \in M} \|x_0 - x\| > 0$

$\exists\ \ell \in X^*$ w/ $\quad \ell|_M \equiv 0$ and $\|\ell\|_{X^*} = 1$

$\ell(x_0) = d$

Pf.



Set $M_0 = M \oplus \operatorname{span}_{\mathbb{R}}\{x_0\}$ whiteboard in my picture

$= \{x + t x_0 : x \in M,\ t \in \mathbb{R}\}$

want to define $f : M_0 \to \mathbb{R}$ satisfying our requirements, in part. $f|_M = 0$ $\ f(x_0) = d$

Ansatz: $\quad f(x + t x_0) = td$

check $\quad \boxed{\|f\|_{L(M_0;\mathbb{R})} = 1}$

If so, then by HB $\exists\ F : X \to \mathbb{R}$ linear $\|F\|_{L(X,\mathbb{R})} = 1$ separating

Prove the two inequalities:

- pick $y = x + t x_0 \in M_0$ so

$$|f(y)| = |t|\, d \le |t| \, \Big\| x_0 - \overbrace{\left(\frac{-x}{t}\right)}^{\text{a point on } M} \Big\|_X$$

$$= \| t x_0 + x \|_X = \|y\|_X$$

$$\implies \quad \|f\|_{L(M_0,\mathbb{R})} \le 1$$

- conversely, given $\epsilon > 0$ $\exists\, x_\epsilon$ in M w/ $d \le \|x_0 - x_\epsilon\|_X < d + \epsilon$

$$|f(x_0 - x)| = d \ge \frac{d}{d+\epsilon} \|x_0 - x_\epsilon\|_X$$

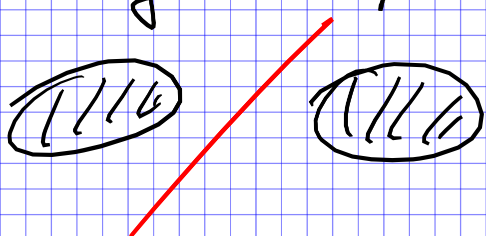
$$\text{so} \quad \|f\|_{L(M_0,\mathbb{R})} \ge \frac{d}{d+\epsilon}$$

$$\text{let } \epsilon \to 0 \qquad \|f\|_{L(M_0,\mathbb{R})} \ge 1 \qquad \blacksquare$$

~ o ~

Rmk. 1 and 2 are just special incarnations of a more general result on the separation of disjoint convex sets

stay tuned.



~ o ~

Annihilator of a set: setup as above, $(X, \|\cdot\|_X)$ normed

Def: Given $A \subset X$ subset (not vec. subspace) we define Annihilator of A

$$A^\perp = \{ f \in X^*, \ f|_A = 0 \}$$

will make sense in Hilbert space setting

rmk. because can reduce to linear subspaces $A^\perp = \operatorname{Span}(A)^\perp$

$(f \in X^* \rightsquigarrow f \text{ is linear})$

Collection of facts about annihilators:

Fact 1:
$$X^\perp = \{0\} \in X^*$$
$$\{0\}^\perp = X^*$$

Fact 2:
$$\overline{M}^\perp = M^\perp \qquad \forall M \subset X \text{ subspace}$$

Pf. $M \subset \overline{M} \implies M^\perp \supset \overline{M}^\perp$

Conversely, we want to check $M^\perp \subset \overline{M}^\perp$

true by continuity: pick $\ell \in M^\perp$ i.e. $\ell(x_0) = 0 \; \forall x_0 \in M$
$$\hookrightarrow \ell(x_0) = 0 \quad \forall x_0 \in \overline{M}$$
$$\implies M^\perp \subset \overline{M}^\perp$$

Two inclusions $\implies M^\perp = \overline{M}^\perp$. ∎

Fact 3: (characterization of the closure)

$M \subset X$ linear subspace, $x_0 \in X$. TFAE:

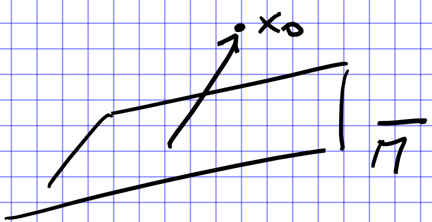
i) $x_0 \in \overline{M}$

ii) $\forall f \in M^\perp$: $f(x_0) = 0$

Pf. i) $\implies$ ii) by continuity, as above: $\underset{\in \overline{M}}{x_0} = \lim_k \underset{\in M}{x_k}$

$$f \in M^\perp \Rightarrow f(x_k) = 0 \; \forall k \xrightarrow[\text{cont.}]{} f(x_0) = 0.$$

ii) $\implies$ i) [contrapositive]) if $x_0 \notin \overline{M}$ then by 2



$$\exists \, \ell \in X^* \quad \ell|_{\overline{M}} = 0$$
$$\ell(x_0) > 0$$

So $\ell \in \overline{M}^\perp \implies \ell \in M^\perp$

but $\ell(x_0) \neq 0$

This is the negative of ii) ∎

Fact 4: $M \subset X$ be a linear subspace. TFAE:

a) $\overline{M} = X$ (M is dense in X)

b) $M^{\perp} = \{0\}$

characterization of dense subspaces

rmk. as a reality check, let's review this equivalence in Hilbert spaces:

a) $\Rightarrow$ b) if $\overline{M} = X \rightsquigarrow \overline{M}^{\perp} = X^{\perp} = \{0\}$

$$M^{\perp} \;\Rightarrow\; b)$$

b) $\Rightarrow$ a) if $M^{\perp} = \{0\} \rightsquigarrow (M^{\perp})^{\perp} = \{0\}^{\perp}$

$$\overline{M} \qquad X \;\Rightarrow\; a)$$

Pf: a) $\Rightarrow$ b) use Fact 1 and Fact 2 above

if $\overline{M} = X \rightsquigarrow \overline{M}^{\perp} = X^{\perp}$

Fact 2 $\|$ Fact 1 $\|$

$M^{\perp} \qquad \{0\} \Rightarrow b)$.

b) $\Rightarrow$ a) use Fact 3 above

if $M^{\perp} = \{0\}$ so $\forall\, x_0 \in X$ part ii) in Fact 3 is TRUE

so $\forall\, x_0 \in X$ part i) in Fact 3 is TRUE

i.e. $\overline{M} \supset X$

but of course $\overline{M} \subset X$ $\Big\}$ $\overline{M} = X$ (a)).

(Dual) Annihilator: setting above $(X, \|\cdot\|_X)$

$(X^*, \|\cdot\|_{X^*})$

given $L \subset X^*$ subset. Define

$$^\perp L = \{ x \in X : \forall \ell \in L \quad \ell(x) = 0 \}$$

dual annihilator

Basics:

- one can reduce to subspaces
- $^\perp L \subset X$ is itself a linear subspace
- replacements for Facts 1, 2, 3, 4 above

in part. $\forall M \subset X$ (linear) subspace have

$$^\perp(M^\perp) = \overline{M}$$

Claim: this identity is just a rephrasing of Fact 3 above.

why? as a specification of the definition of $^\perp L$ above, we have

$$^\perp(M^\perp) = \{ x \in X : \forall \ell \in M^\perp \quad \ell(x) = 0 \}$$