Functional Analysis I - L15 5/11/2020

Duality in Hilbert spaces

what we discussed last time is still true

but here we can say more → new statements

→ more concrete incarnations of "old" facts

Today let H be a Hilbert space over $\mathbb{R}$.

Consider:

$$H \xrightarrow{J} H^*$$
$$y \longmapsto j_y$$

w/ $j_y(x) = (x,y)_H$

<u>Lemma</u>: the map $J: H \to H^*$ is a linear <u>isometry</u> in the sense that

$$\forall y \in H \qquad \|y\|_H = \|j_y\|_{H^*}.$$

In particular, J is <u>injective</u>.

<u>Pf.</u>

$$\|j_y\|_{H^*} = \sup_{\|x\|\le 1} |j_y(x)| = \sup_{\|x\|\le 1} |(x,y)|$$
$$\le \sup_{\|x\|\le 1} \|x\|\,\|y\| \le \|y\|$$

$$\Longrightarrow \quad \|j_y\|_{H^*} \le \|y\|$$

To prove equality, just pick $x := \frac{y}{\|y\|}$ $\quad \|j_y\|_{H^*} \ge \left|j_y\left(\frac{y}{\|y\|}\right)\right| = \|y\|$ ∎

<u>Thm.</u> (Riesz) the map $J: H \to H^*$ is surjective

i.e. $\quad \forall \ell \in H^* \quad \exists!\, y \in H$ s.t. $\quad j_y = \ell$

$$\left(\Longleftrightarrow \quad \ell(x) = (x,y) = j_y(x) \quad \forall x \in H\right).$$

<u>Moral</u>: J allows for an isometric identification of H and H^*

(in certain situations we just identify H and H^*...)

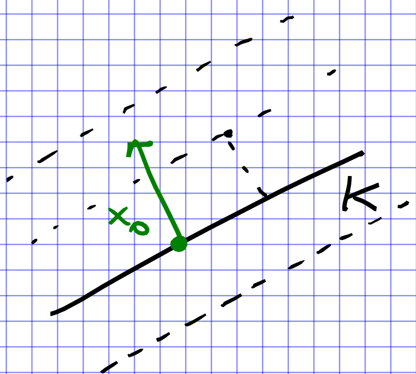
Pf.

If $\ell \equiv 0$ pick $y = 0$ ✓, thus WLOG assume $\ell \not\equiv 0$

Then $K := \ker(\ell)$ is a linear subspace (ℓ is linear)

closed ($K = \ell^{-1}(0)$, $\ell \in H^*$)

and $K \subsetneq H$ because $\ell \not\equiv 0$.



[visually speaking: the functional ℓ is the "height function w.r.t. K"

Since $K \subsetneq H$ by the projection lemma in Hilbert spaces $\exists x_0 \in H \setminus K$ s.t. $x_0 \perp K$. Since $x_0 \notin K$ $\ell(x_0) \neq 0$ so WLOG $\ell(x_0) = 1$.

but now given any $x \in H$ have $\underbrace{x - \ell(x) x_0} \in K$

thus $(x_0, x - \ell(x) x_0) = 0$

$\iff$ $(x_0, x) = \ell(x) \|x_0\|^2 \iff \ell(x) = \Big(\overbrace{\frac{x_0}{\|x_0\|^2}}^{y}, x\Big)$ $\forall x \in H$

Also since J is an isometry $J^{-1}(\ell)$ is a unique element. ∎

Thm. (Lax-Milgram): let $a : H \times H \to \mathbb{R}$ be bilinear and continuous (so that in part. $\exists \Lambda > 0$ s.t. $|a(x,y)| \leq \Lambda \|x\| \|y\|$) [problem 4.5]

Assume $\exists \lambda > 0$ s.t. $\forall x \in H$ $\underbrace{a(x,x) \geq \lambda \|x\|^2}_{\text{coercivity}}$ (for bilinear maps)

Then $\exists A \in L(H) = L(H,H)$ w/

$$\boxed{a(x,y) = (Ax, y)} \quad \forall x, y \in H$$

Also, $\|A\|_{L(H)} \leq \Lambda$, $\|A^{-1}\|_{L(H)} \leq \lambda^{-1}$.

Rmk: the form a above is not nec. symmetric.

Rmk. : all coercive cont. bilinear forms are a "modified" scalar product via a linear continuous operator.

Corollary (generalised Riesz rep. theorem)

Given $(H, (\cdot,\cdot))$ and $\underset{\smile}{a}$ as above, then
$\forall\ \ell \in H^*\ \ \exists\ y \in H$ w/ $\boxed{\ell(x) = a(y,x)}$ $\forall x \in H$.
Furthermore $\|y\| \leq \lambda^{-1} \|\ell\|_{H^*}$.

Pf. (corollary given thm)

Step 1: $\exists\ A \in L(H)$ w/ $a(y,x) = (Ay,x)$ $\forall x, y \in H$.
by Lax-Milgram

Step 2: $\exists!\ z \in H$ w/ $\ell(x) = (z,x)$ $\forall x \in H$
by Riesz

Step 3: solve for y the eq. $Ay = z$
S1 + S2 + Lax-Milgram
given in step 2

Lax-Milgram gives $A^{-1} \in L(H)$ so $y := A^{-1}z$
Hence $\ell(x) = (z,x) = (Ay,x) = a(y,x)$
Further: $\|y\| = \|A^{-1}z\| \leq \underbrace{\|A^{-1}\|_{L(H)}}_{\leq \lambda^{-1}} \underbrace{\|z\|}_{= \|\ell\|_{H^*}}$ ∎

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Proof (Lax-Milgram)

Step 0: arguement "one point at a time".

Given $x \in H$ consider $\underbrace{\ell_x}: y \longmapsto a(x,y)$
(similar but not to be confused w/ j_y)

- linear
- continuous (i.e. $\ell_x \in H^*$)
  why? $|\ell_x(y)| = |a(x,y)| \leq \Lambda \|x\| \|y\|$ $\rightarrow \|\ell_x\|_{H^*} \leq \Lambda \|x\| < \infty$

So by Riesz $\exists!\ \boxed{\underbrace{z}_{Ax} = J^{-1}\ell_x}$ such that $a(x,y) \equiv \ell_x(y) = (z,y)\ \ \forall y \in H$

$A : H \circlearrowleft$ is a well-defined set-theoretic map

Also A is linear since J is (J is a linear isometry, so is J^{-1}), and continuous because

$$\|Ax\| \underset{\substack{\uparrow \\ J \text{ isometry}}}{=} \|\ell_x\|_{H^*} \le \Lambda \|x\|$$

$$\Longrightarrow \|A\|_{\mathcal{L}(H)} \le \Lambda$$

Step 1: A injective w/ $\underbrace{\|Ax\| \ge \lambda \|x\|}_{\text{coercivity (for a linear map)}}\ \ \forall x \in H$

why? hp. $a(x,x) \ge \lambda \|x\|^2\ \ \forall x \in H$

$$a(x,x) \underset{\substack{\uparrow \\ \text{Step 0}}}{=} (Ax, x) \underset{\substack{\uparrow \\ \text{Cauchy-S.}}}{\le} \|Ax\|\,\|x\|$$

put together

get $\|Ax\| \ge \lambda \|x\|\ \ \forall x \in H$

thus (since $\lambda > 0$) $\Longrightarrow$ A injective w/ estimate above

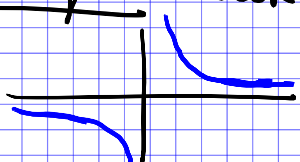
Step 2: image$(A) \equiv A(H)$ is closed (in H).

(**Achtung!** we are not saying that $\underbrace{\text{the graph } \Gamma_A \text{ is closed in } H \times H}_{\text{this is TRUE since } A \text{ is continuous}}$

BUT Γ_A closed $\not\Longrightarrow$ image$(A) \subset H$ is closed.

Indeed, the projections $\pi : H \times H \longrightarrow H$ e.g. on 2nd factor are not closed in general i.e. they do not have the property that "image of closed sets are closed"

Example: take Γ in $\mathbb{R}^2$ to be the hyperbola $1/x$



$\Gamma \subset \mathbb{R}^2$ closed but $\underbrace{\pi(\Gamma)}_{\mathbb{R} \setminus \{0\}} \subset \mathbb{R}$ is not closed)

That said, let's give a tour of proof of the claim above:

pick a sequence (x_n) s.t. $\boxed{Ax_k \longrightarrow y \in H}$

? $\quad y \overset{?}{\in} \text{image}(A)$

$$\|x_k - x_\ell\| \underset{\text{Step 1}}{\le} \lambda^{-1}\|A(x_k - x_\ell)\| = \lambda^{-1}\|Ax_k - Ax_\ell\|$$

(x_n) is also Cauchy! $\qquad\qquad (Ax_k)$ is Cauchy

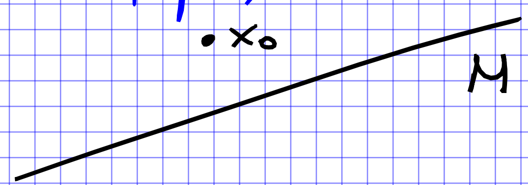
hence, by completeness, $\quad x_k \longrightarrow x$

by cont. $\quad Ax_k \longrightarrow Ax$, $\quad Ax_k \longrightarrow y$ $\quad\Big\}$ uniqueness of the limit $\Rightarrow y = Ax$ ∎

Step 3: A is surjective.

By contradiction, suppose $\underbrace{\text{image}(A) = M}_{\text{closed linear subspace}} \subsetneq H$ (proper)

Then
- $\exists\, x_0 \in H \setminus M$
- $\exists\, \ell : H \to \mathbb{R}$ $\quad (\ell \in H^*)$



$$\ell|_M \equiv 0$$

$$\ell(x_0) = d := \text{dist}(x_0, M)$$

by Riesz $\quad y := J^{-1}\ell \quad (y \neq 0 \text{ in } H)$

$$0 < \lambda\|y\|^2 \le a(y,y) \underset{\text{Step 0}}{=} (Ay, y) \underset{\boxed{y = J^{-1}\ell}}{=} \ell(\overbrace{Ay}^{\in M}) = 0$$

This contradiction shows that $\text{image}(A) \equiv H$ (i.e. A surjective).

- **Conclusion:** open mapping theorem $\quad \exists\, A^{-1} \in L(H)$

For the norm estimate: given $x \in H$

$$\|A^{-1}x\| \le \lambda^{-1}\|A(A^{-1}x)\| \equiv \lambda^{-1}\|x\|$$

coercivity estimate for A, proven in Step 1 $\quad \Longrightarrow \|A^{-1}\|_{L(H)} \le \lambda^{-1}$ ∎