Functional Analysis I - L16 9/11/2020

Duality for L^p spaces

Thm.: Let $\Omega \subseteq \mathbb{R}^n$ be open ($\neq \emptyset$) and let μ denote a non-negative Radon measure on Ω. Then given $1 \leq p < \infty$ we have that $(L^p(\Omega))^*$ is isometrically isomorphic to $L^q(\Omega)$ w/ $\frac{1}{p} + \frac{1}{q} = 1$. (More explicitly, there exists a surjective isometry $J : L^q(\Omega) \longrightarrow (L^p(\Omega))^*$.) ✓

Rmk. conclusion is false if $p = \infty$, in fact $(L^\infty(\Omega))^* \supsetneq L^1(\Omega)$

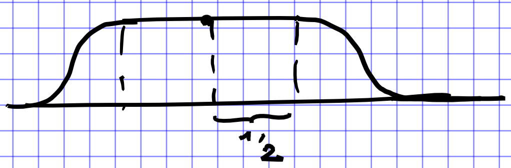
why? Ω as above, $\underbrace{\mu = \mathcal{L}^n.}_{\text{for } x_0 \in \Omega}$ Pick the functional

$$\delta_{x_0} : C^0(\bar{\Omega}) \longrightarrow \mathbb{R} \qquad \text{bounded, linear map}$$
$$f \longmapsto f(x_0) \qquad |\delta_{x_0}(f)| \leq \|f\|_{C^0}$$

so by Hahn-Banach $\exists$ extension $\ell : L^\infty(\Omega) \longrightarrow \mathbb{R}$ bounded, linear operator.

Suppose by contradiction that $\ell = \ell_g$ for some $g \in L^1(\Omega)$ where $\ell_g(f) = \int_\Omega f g \, d\mu$. WLOG say $x_0 = 0$ $B_1(\underline{0}) \subset \Omega$

Consider $\varphi \in C_c^\infty(B_1(\underline{0}))$, $0 \leq \varphi \leq 1$ $\varphi \equiv 1$ on $B_{1/2}(\underline{0})$



rescale it: $\varphi_k(x) = \varphi(kx)$

then $\forall k \in \mathbb{N}_*$

$$1 = \delta_0(\varphi_k) = \ell(\varphi_k) = \int_\Omega g \varphi_k \xrightarrow[k \to \infty]{} 0$$
(dominated convergence)

Contradiction, this shows that $(L^\infty(\Omega))^*$ contains more stuff than just L^1. In fact ↗ is "a space of measures" (ref: Riesz-Markov rep. theorem).

Advice: review rmk. above after proof of the theorem.

Proof: given $g \in L^q(\Omega)$, consider $\ell_g(f) = \int_\Omega f g \, d\mu$

linear, by Hölder $|\ell_g(f)| \le \|f\|_{L^p} \|g\|_{L^q}$

so $\|\ell_g\|_{(L^p)^*} \le \|g\|_{L^q}$. Moral: there is a well-defined linear map $J : L^q \longrightarrow (L^p)^*$ w/ $\|J\|_{\mathcal{L}(L^q;(L^p)^*)} \le 1$.

Step 1: check of isometry property. Two cases:

i) $1 < p < \infty$ so $\frac{1}{p} + \frac{1}{q} = 1 \iff p = \frac{q}{q-1} \iff p(q-1) = q$

hence (recall equality case for Hölder)

pick $f = g|g|^{q-2}$ so that $|f|^p = |g|^q$

thus $f \in L^p$ and $\|f\|^p_{L^p} = \int_\Omega |f|^p = \int_\Omega |g|^q$

• i.e. $\|f\|_{L^p} = \|g\|^{q-1}_{L^q}$

•• $\ell_g(f) = \int_\Omega |g|^q = \|g\|^q_{L^q} \le \|\ell_g\|_{(L^p)^*} \|f\|_{L^p} = \|\ell_g\|_{(L^p)^*} \|g\|^{q-1}_{L^q}$

so $\boxed{\|\ell_g\|_{(L^p)^*} \ge \|g\|_{L^q}}$

Conclusion: $\|\ell_g\|_{(L^p)^*} = \|g\|_{L^q}$

ii) $\boxed{p = 1}$ ($\iff q = \infty$) so we have $g \in L^\infty$; given $\epsilon > 0$

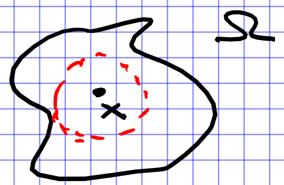
pick $x \in \Omega$ s.t.

(*) $\lim_{r \to 0} \left| \frac{1}{\mu(B_r(x))} \int_{B_r(x)} g(y) \, d\mu(y) \right| \ge \|g\|_{L^\infty} - \epsilon$

why? $\longrightarrow$ definition of L^∞ norm

$\longrightarrow$ Lebesgue thm. for "continuity in mean"

now fix $r_* > 0$, $r_* < \mathrm{dist}(x, \partial\Omega)$



w/ $\left| \frac{1}{\mu(B_{r_*}(x))} \int_{B_{r_*}(x)} g \, d\mu \right| \ge \|g\|_{L^\infty} - 2\epsilon$

pick $\quad f = \frac{1}{\mu(B_{r_*}(x))} \chi_{B_{r_*}(x)} \in L^1(\Omega)$

w/ $\|f\|_{L^1} = 1$

$$\left| l_g(f) \right| = \left| \frac{1}{\mu(B_{r_*}(x))} \int_{B_{r_*}(x)} g \, d\mu \right| \geq \|g\|_{L^\infty} - 2\epsilon$$

let $\epsilon \to 0^+ \implies \boxed{\| l_g \|_{(L^1)^*} \geq \|g\|_{L^\infty}}$

Conclusion: equality must hold, i.e. $J : L^q \to (L^p)^*$ is an isometry.

Real issue: surjectivity (i.e. we must prove that every element of $(L^p)^*$ can be represented in this fashion, namely by integration against an element of L^q).

We'll now prove surjectivity, starting w/ $\boxed{1 < p < \infty}$

for this, we need the following lemma:

Lemma (1st Clarkson inequality) Given $1 < p < \infty$

let $u, v \in L^p(\Omega)$ w/ $\|u\|_{L^p} = \|v\|_{L^p} = 1$ and set $\epsilon := \|v - u\|_{L^p} > 0$. Then:

i) for $2 \leq p < \infty$ $\quad \left\| \frac{u+v}{2} \right\|_{L^p} \leq \left(1 - \frac{\epsilon^p}{2^p} \right)^{1/p}$

ii) for $1 < p \leq 2$ $\quad \left\| \frac{u+v}{2} \right\|_{L^p} \leq \left(1 - \frac{\epsilon^q}{2^q} \right)^{1/q}$

Proof (lemma): we'll only show i), for ii) see Adams.

- $\alpha^p + \beta^p \leq (\alpha^2 + \beta^2)^{p/2} \quad \forall \alpha, \beta > 0$

[by p-homogeneity, can assume WLOG $\beta = 1$, but

$f(x) = (x^2 + 1)^{p/2} - x^p - 1 \nearrow$ on $[0, \infty)$

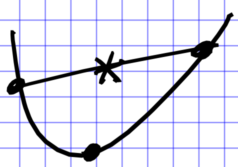
w/ $f(0) = 0$ so $f(x) \geq 0$]

- just pick $\alpha = \left| \frac{a+b}{2} \right| \quad \beta = \left| \frac{a-b}{2} \right| \quad a, b \in \mathbb{R}$

By the previous inequality:

$$\left|\frac{a+b}{2}\right|^p + \left|\frac{a-b}{2}\right|^p \le \left(\left|\frac{a+b}{2}\right|^2 + \left|\frac{a-b}{2}\right|^2 \right)^{p/2}$$

$$= \left(\frac{a^2+b^2}{2}\right)^{p/2}$$



$$\le \frac{1}{2}\left((a^2)^{p/2} + (b^2)^{p/2} \right)$$

by convexity of $t \longmapsto t^{p/2}$ (for $p \ge 2$)

$$= \frac{1}{2}\left(|a|^p + |b|^p \right)$$

- take $a = u(x)$ $b = v(x)$, specialise ineq. above and integrate over Ω. We get:

$$\left\| \frac{u+v}{2} \right\|_{L^p}^p \le \frac{1}{2}(1+1) - \left\| \frac{u-v}{2} \right\|_{L^p}^p$$

$$= 1 - \frac{\epsilon^p}{2^p} \qquad (\epsilon = \|v-u\|_{L^p})$$

$$\left\| \frac{u+v}{2} \right\|_{L^p} \le \left(1 - \frac{\epsilon^p}{2^p}\right)^{1/p}$$

□

<u>Step 2.1</u> (surjectivity of J, $1<p<\infty$)

Given $\ell \in (L^p(\Omega))^*$ wlog $\ell \not\equiv 0$ (else pick $g \equiv 0$)

by scaling wlog $\|\ell\|_{(L^p)^*} = 1$. Pick a <u>maximising</u> seq. for the operator norm of ℓ, i.e.

$$(f_k) \subset L^p(\Omega) \quad w/ \quad \|f_k\|_{L^p} = 1$$

$$\ell(f_k) \nearrow \|\ell\|_{(L^p)^*} = 1$$

<u>Claim</u>: (f_k) is a Cauchy sequence in $L^p(\Omega)$.

check claim:

$$1 + o(1) = \frac{1}{2}\left(\ell(f_h) + \ell(f_k)\right) = \frac{1}{2}\left(\ell(f_h+f_k)\right)$$

$$\le \underbrace{\|\ell\|_{(L^p)^*}} \left\| \frac{f_h+f_k}{2} \right\|_{L^p} \underset{\text{Clarkson}}{\le} \underbrace{\|\ell\|_{(L^p)^*}}_{=1} \left(1 - \frac{\|f_h-f_k\|_{L^p}^p}{2^p}\right)^{1/p}$$

So (by completeness) $f_k \longrightarrow f$ in $L^p(\Omega)$

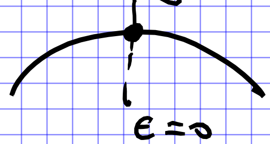
w/ $\|f\|_{L^p} = 1$, $\ell(f) = \lim_{k\to\infty} \ell(f_k)$

$$= 1$$

Now we use a variational argument (related to 1^{st} derivative test for f, which is a maximizer...) to prove that $\ell = \ell_g$ with $g = f\,|f|^{p-2} \in L^q(\Omega)$.

Set $f_\epsilon := \dfrac{f+\epsilon\varphi}{\|f+\epsilon\varphi\|_{L^p}}$ for $\varphi \in L^p(\Omega)$

For any fixed φ the $\epsilon \longmapsto \ell(f_\epsilon)$ has a max at $\epsilon=0$



$$\rightsquigarrow \boxed{\left(\frac{d}{d\epsilon}\right)_{\epsilon=0} \ell(f_\epsilon) = 0}$$ Euler-Lagrange equation

Compute the derivative

$$\left(\frac{d}{d\epsilon}\right|_{\epsilon=0} \ell(f_\epsilon) = \frac{\ell(\varphi) - \ell(f)\left(\frac{d}{d\epsilon}\right)_{\epsilon=0}\|f+\epsilon\varphi\|_{L^p}}{\|f\|^2_{L^p}}$$

recall $\|f\|_{L^p} = 1$

$$\left(\frac{d}{d\epsilon}\right|_{\epsilon=0} \left(\int_\Omega |f+\epsilon\varphi|^p\right)^{1/p} = \frac{1}{p}\underbrace{\left(\int_\Omega |f+\epsilon\varphi|^p\right)^{1/p-1}\Bigg|_{\epsilon=0}}_{=1} \times$$

$$\times \left(\frac{d}{d\epsilon}\right|_{\epsilon=0} \int_\Omega |f+\epsilon\varphi|^p = \frac{1}{p}\left(\frac{d}{d\epsilon}\right|_{\epsilon=0} \int_\Omega |f+\epsilon\varphi|^p$$

[trick: $|f+\epsilon\varphi|^p = \left(|f+\epsilon\varphi|^2\right)^{p/2}$

$$= \frac{1}{p}\int_\Omega p\,|f|^{p-2} f\varphi\, d\mu$$

$$= \int_\Omega \underbrace{f\,|f|^{p-2}}_{\in L^q}\varphi\, d\mu$$

So if we set $g = f\,|f|^{p-2} \in L^q(\Omega)$

we get (1^{st} der. test)

$$0 = \ell(\varphi) - \underbrace{\ell(f)}_{=1}\int_\Omega g\varphi\, d\mu \iff \boxed{\begin{array}{l}\ell(\varphi) = \int_\Omega g\varphi\, d\mu \\ \forall \varphi \in L^p(\Omega)\end{array}}$$

We proved surjectivity for $1<p<\infty$. Argument above used <u>Clarkson</u> which is only true for $1<p<\infty$. Thus $\boxed{p=1}$ needs a different proof.

<u>Step 2.2</u> (surjectivity of J, for $\boxed{p=1}$)

<u>Lemma</u>: (setting above...), assume further $\mu(\Omega)<\infty$ (finite measure).

Then

" $\varphi \in \bigcap_{p \geq p_0} L^p$ w/ uniform estimate $\Longrightarrow \varphi \in L^\infty$ "

in fact $\boxed{\|\varphi\|_{L^\infty} \leq \liminf_{p\to\infty} \|\varphi\|_{L^p}}$

(actually: $\|\varphi\|_{L^\infty} = \lim_{p\to\infty} \|\varphi\|_{L^p}$)

<u>Pf.</u> pick $a \in \mathbb{R}_{>0}$ such that $\mu(|\varphi|>a)>0$

$$\mu(|\varphi|>a) = \mu(|\varphi|^p > a^p) \underset{\text{Markov}}{\leq} \frac{1}{a^p}\int_\Omega |\varphi|^p d\mu = \frac{1}{a^p}\|\varphi\|_{L^p}^p$$

$$\Longleftrightarrow \quad a\left(\mu(|\varphi|>a)\right)^{1/p} \leq \|\varphi\|_{L^p}$$

$p\to\infty$ $\quad a \leq \liminf_{p\to\infty} \|\varphi\|_{L^p}$

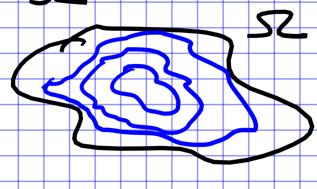
I can now let $a \to \operatorname{ess\,sup}|\varphi| = \|\varphi\|_{L^\infty}$

$$\|\varphi\|_{L^\infty} \leq \liminf_{p\to\infty} \|\varphi\|_{L^p}$$ ■

Now, let's take care of Step 2.2 i.e. surjectivity of J for $p=1$.

We assume (for simplicity) $\mu(\Omega)<\infty$, else need an approximation argument using an exhaustion of Ω

Ω



if $s \geq 1$ and $\mu(\Omega) < \infty$ $\quad \boxed{L^s(\Omega) \xhookrightarrow{i_s} L^1(\Omega)}$

$$\|i_s\|_{L(L^s, L^1)} \leq \mu(\Omega)^{1-1/s} \longrightarrow 1 \quad (s \to 1)$$

($\underline{\text{why}}$? by Hölder $\int |f| \, d\mu \leq \|f\|_{L^s} \, \mu(\Omega)^{1-1/s}$

$$\frac{\|f\|_{L^1}}{\|f\|_{L^s}} \leq \mu(\Omega)^{1-1/s} \;)$$

reverse inclusion at the level of duals, i.e.

$$(L^1(\Omega))^* \subset (L^s(\Omega))^*$$

$\underline{\text{why?}}$

$$\begin{tikzcd} L^s(\Omega) \arrow[r, "i_s"] \arrow[dr, dashed, "\ell \circ i_s"'] & L^1(\Omega) \arrow[d, "\ell"] \\ & \mathbb{R} \end{tikzcd}$$

given $\ell \in (L^1(\Omega))^*$ we consider $\ell \circ i_s \in (L^s(\Omega))^*$

w/ $\boxed{\limsup_{s \to 1} \|\ell\|_{(L^s(\Omega))^*} \leq \|\ell\|_{(L^1(\Omega))^*}}$

$\underline{\text{why?}}$

$$\|\ell\|_{(L^1(\Omega))^*} = \sup_{\substack{\varphi \in L^1 \\ \|\varphi\|_{L^1} \leq 1}} |\ell(\varphi)|$$

$$\geq \sup_{\substack{\varphi \in L^s(\Omega) \\ \|i_s(\varphi)\|_{L^1} \leq 1}} |\ell(\varphi)| \geq \boxed{\sup_{\substack{\varphi \in L^s(\Omega) \\ \|\varphi\|_{L^s} \leq \|i_s\|^{-1}_{L(L^s,L^1)}}} |\ell(\varphi)|}$$

$$\|i_s(\varphi)\|_{L^1} \leq \|i_s\|_{L(L^s,L^1)} \|\varphi\|_{L^s}$$

$$= \sup_{\substack{\psi \in L^s \\ \|\psi\|_{L^s(\Omega)} \leq 1}} \underbrace{\|i_s\|^{-1}_{L(L^s,L^1)}}_{\text{does not depend on } \psi} |\ell(\psi)| = \|i_s\|^{-1}_{L(L^s,L^1)} \|\ell\|_{(L^s(\Omega))^*}$$

$$\psi = \varphi \, \|i_s\|_{L(L^s,L^1)}$$

Take the inequality we proved i.e. $\|\ell\|_{(L^1(\Omega))^*} \geq \|i_s\|^{-1}_{L(L^s,L^1)} \|\ell\|_{(L^s(\Omega))^*}$ and send $s \to \infty$ to get the claim.

Given $\ell \in (L^1(\Omega))^*$, $\ell = \ell \circ i_s \in L^s(\Omega)^*$

$$\overset{\Longrightarrow}{\text{Step 2.1}} \quad \ell = \ell_g \text{ w/ } g = g_r \in L^r(\Omega)$$

w/ $\frac{1}{r} + \frac{1}{s} = 1$

hence $\ell(\varphi) = \ell_{g_r}(\varphi) = \int_\Omega g_r \varphi \, d\mu$

$\forall \varphi \in C_c^\infty(\Omega)$

Pick $1 < s < s'$ duals $r' < r < \infty$ have $\forall \varphi \in C_c^\infty(\Omega)$

$$\underbrace{\ell_{g_r}(\varphi) = \ell_{g_{r'}}(\varphi)}_{\ell(\varphi)} \Longrightarrow \int_\Omega g_r \varphi \, d\mu = \int_\Omega g_{r'} \varphi \, d\mu$$

$$\Longleftrightarrow \int_\Omega (g_r - g_{r'}) \varphi \, d\mu = 0 \quad \forall \varphi \in C_c^\infty(\Omega)$$

lemma (fund. lemma of Calc. Var.) $\Longrightarrow \boxed{g_r = g_{r'}}$

This is true for any given couple $r' < r < \infty$ thus

$$g := g_r \ (= g_{r'}) \in \bigcap_{r<\infty} L^r(\Omega) \quad \text{w/}$$

$$\| g \|_{L^r} = \| g_r \|_{L^r} \underset{\text{Step 1/2.1}}{=} \| \ell_{g_r} \|_{(L^s)^*} = \| \ell \|_{(L^s)^*}$$

so
$$\underbrace{\limsup_{r \to \infty} \| g \|_{L^r}}_{} = \limsup_{s \to 1} \| \ell \|_{(L^s)^*}$$

$$\underset{\text{above}}{\leq} \| \ell \|_{(L^1)^*}$$

by the lemma $\quad g \in L^\infty$ w/ $\| g \|_{L^\infty} \leq \| \ell \|_{(L^1)^*}$

so $(p = 1)$ $\quad \ell = \ell_g$ for $\uparrow$ $\Longrightarrow J$ surjective ∎